Modelling Mortality Using Multiple Stochastic Latent Factors*

Jorge Miguel Bravo†

Preliminary version: May 2009

Abstract

In this paper we develop a new model for stochastic mortality that considers the possibility of both positive and negative catastrophic mortality shocks. Specifically, we assume that the mortality intensity can be described by an affine function of a finite number of latent factors whose dynamics is represented by affine-jump diffusion processes. The model is then embedded into an affine-jump framework, widely used in the term structure literature, in order to derive closed-form solutions for the survival probability. This framework and model application to the classical Gompertz-Makeham mortality law provides a theoretical foundation for the pricing and hedging of longevity-linked derivatives.

JEL Code: G22

Keywords: stochastic mortality intensity; longevity risk; affine-jump models.

*Paper prepared for presentation at the **7th International Workshop on Pension, Insurance and Saving**, Paris, 28-29 May, 2009.

†University of Évora - Department of Economics and CEFAGE-UE / CIEF, Largo dos Colegiais, N.º 2, 7000-803, Évora/Portugal, Tel.: +351 266 740894, E-Mail: jbravo@uevora.pt.

1 Introduction

Longevity risk, i.e., the risk that members of some reference population might live longer, on average, than anticipated, has recently emerged as one of the largest sources of risk faced by life insurance companies, pension funds, annuity providers, life settlement investors and a number of other potential players in the marketplace for this risk. For instance, given the uncertainty about future developments in mortality and life expectancy, pension funds and annuity providers run the risk that the net present value of their pension promises and annuity payments will turn out to be higher than expected, as they will have to pay out a periodic sum of income that will last for an uncertain life span.

This risk is amplified by the current problems in state-run pay-as-you-go social security systems, by the market trend away from defined-benefit corporate pension schemes towards defined-contribution plans and by the increasing instability and mobility in the labour market that breaks down traditional family networks. In this environment, individuals will have to become more self-reliant and will wish to diversify their sources of income in retirement, assigning in particular a greater weight to private solutions, namely annuities and other more complex longevity-linked securities.

One of the key conditions for the development of longevity-linked products and markets and for the hedging of longevity risk is the development of generally agreed market models for risk measurement. Historically, actuaries have been calculating premiums and mathematical reserves using a deterministic approach, by considering a deterministic mortality intensity, which is a function of the age only, extracted from available (static) lifetables and by setting a flat ("best estimate") interest rate to discount cash flows over time. Since neither the mortality intensity nor interest rates are actually deterministic, life insurance companies are exposed to both financial and mortality (systematic and unsystematic) risks when pricing and reserving for any kind of long-term living benefits.

In order to protect the company from mortality improvements, actuaries have different solutions, among them to resort to projected (dynamic or prospective) lifetables, i.e., lifetables including a forecast of future trends of mortality instead of static lifetables. For their construction, a number of different discrete-time projection models have been proposed and are actually used in actuarial practice. Tuljapurkar and Boe (1998), Tabeau (2001), GAD (2001), Pitacco (2004), Wong-Fupuy and Haberman (2004), Booth (2006) and Bravo (2007) provide a detailed review of historical patterns in mortality and longevity forecasting models.

Since the future mortality is actually unknown, there is a likelihood that future death rates will turn out to be different from what we've projected, and so a better assessment of mortality and longevity risks would be one that consists of both a mean estimate and a measure of uncertainty. Such assessment can only be performed by using stochastic models to describe both demographic and financial risks.

Up to now, a number of different stochastic mortality models have been proposed (for a detailed classification see, e.g., Cairns *et al.* (2006a) and Bravo (2007)). Most of these stochastic mortality models are *short rate mortality models*, i.e., they model the spot mortality rate $q_x(t)$, or the spot force of mortality $\mu_x(t)$. Milevsky and Promislow (2001) were the first to propose a stochastic "hazard rate" or force of mortality. With the intention of pricing guaranteed annuitization options in variable annuities, the authors demonstrate, first in a discrete-time framework, how to price and hedge a plain vanilla mortality option using a portfolio composed by zero coupon bonds, insurance contracts and endowment contracts. Moreover, they price the same option in a continuous-time risk-neutral framework assuming that the dynamics of the short interest rate and of the mortality intensity evolve independently over time according to a Cox-Ingersoll-Ross-process and a stochastic *mean reverting Brownian Gompertz-type* model, respectively.

Dahl (2004) develops a general stochastic model for the mortality intensity. The author derives partial differential equations for both the price at which some insurance contracts should be sold on the financial market and for the general mortality derivatives in the presence of stochastic mortality. In addition, he envisages solutions by which systematic mortality risk can be transferred to the financial market. Dahl and Moller (2005) derive risk-minimizing strategies for insurance liabilities in a market without derivative securities. Biffis and Millossovich (2004) expand this framework to a bidimensional setting in order to deal effectively with several sources of risk that simultaneously affect insurance contracts.

In Biffis (2005), affine jump-diffusion processes are used to model both financial and demographic factors. Specifications of the model with an affine term structure are employed and closed form mathematical expressions (up to the solutions of standard Riccati ordinary differential equations) are derived for some classic life insurance contracts. Biffis and Denuit (2005) and Biffis *et al.* (2006) generalize the model proposed by Lee and Carter (1992) to a stochastic setting. The authors assume that the dynamics of the time-varying parameter κ_t can be described by stochastic differential equations.

Most stochastic mortality models presented up to now exhibit three main limitations. First, they are single factor models in that the assume that mortality shocks afect all ages and cohorts in the same way. Second, they have generally been implemented for single age cohorts. Third, they underestimate the importance of jump movements in explaining mortality dynamics over time.

In Schrager (2006) the author first addresses these problems by presenting an affine stochastic mortality model that simultaneously describes the evolution of mortality for different age groups as opposed to the previous formulation in which a single cohort is considered. The author fits the model to Dutch mortality data using Kalman filters and presents alternative valuation approaches for a number of mortality-contingent contracts.

In this paper we expand the approach proposed by Schrager (2006) by developing a new model for stochastic mortality that considers the possibility of both positive and negative catastrophic mortality shocks. Specifically, we assume that the mortality intensity can be described by an affine function of a finite number of latent factors whose dynamics is represented by affine-jump diffusion processes. The model is then embedded into an affine-jump framework, widely used in the term structure literature, in order to derive closed-form solutions for the survival probability. This framework and model application provides a theoretical foundation for the pricing and hedging of longevity-linked derivatives.

The paper is organized as follows. In Section 2 we develop the mathematical framework for stochastic mortality used throughout the paper. In Section 3 we illustrate the use of this approach by revisiting the classical Gompertz-Makeham mortality law. Finally, Section 4 concludes.

2 Affine-Jump diffusion processes for mortality

To model mortality we follow the standard approach and draw a parallel between insurance contracts and certain credit-sensitive securities and exploit some results of the intensity-based approach to credit risk modelling. Specifically, we use doubly stochastic processes (also known as *Cox processes*) in order to model the random evolution of the stochastic force of mortality of an individual aged x in a manner that is common in the credit risk literature.

We are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and concentrate on an individual aged x at time 0. Following the pioneering work of Artzner and Delbaen (1995) in the credit risk literature and the proposals by Dahl (2004) and Biffis (2005) among others in the mortality area, we model his/her random lifetime as an $\mathbb{F}$-stopping time τ_x admitting a random intensity μ_x. Specifically, we consider τ_x as the first jump-time of a nonexplosive $\mathbb{F}$-counting process N recording at each time $t \geq 0$ whether the individual has died $(N_t \neq 0)$ or survived $(N_t = 0)$. The stopping time τ_x is said to admit an intensity μ_x if the compensator of N does, i.e., if μ_x is a nonnegative predictable process such that $\int_0^t \mu_x(s)ds < \infty$ for all $t \geq 0$ and such that the compensated process $M_t = \left\{ N_t - \int_0^t \mu_x(s)ds : t \geq 0 \right\}$ is a local $\mathbb{F}$-martingale. If the stronger condition $\mathbb{E}\left(\int_0^t \mu_x(s)ds \right) < \infty$ is satisfied, then M_t is an $\mathbb{F}$-martingale.

From this, we derive

$$\mathbb{E}\left(N_{t+\Delta t} - N_t \middle| \mathcal{F}_t \right) = \mathbb{E}\left(\int_t^{t+\Delta t} \mu_x(s)ds \middle| \mathcal{F}_t \right), \tag{1}$$

based on which we can write

$$E\left(N_{t+\Delta t} - N_t \middle| \mathcal{F}_t \right) = \mu_x(t)\Delta t + o\left(\Delta t\right), \tag{2}$$

an expression comparable with that of the instantaneous probability of death ${}_{\Delta t}q_{x+t}$ derived in the traditional deterministic context.

By further assuming that N is a Cox (or doubly stochastic) process driven by a subfiltration $\mathbb{G}$ of $\mathbb{F}$, with $\mathbb{F}$-predictable intensity μ it can be shown, by using the law of iterated expectations, that the probability of an individual aged $x + t$ at time t surviving up to time $T \geq t$, on the set $\{\tau > t\}$, is given by

$$\mathbb{P}\left(\tau > T \middle| \mathcal{F}_t \right) = \mathbb{E}\left[e^{-\int_t^T \mu_{x+s}(s)ds} \middle| \mathcal{F}_t \right]. \tag{3}$$

Readers who are familiar with mathematical finance and, in particular, with the interest rate literature, can without difficulty observe that the right-hand-side of equation (3) represents the price at time t of a unitary default-free zero coupon bond with maturity at time $T > t$, if the intensity μ is to represent the short-term interest rate.

One of the main advantages of this mathematical framework is that we can approach the survival probability (3) by using well known affine-jump diffusion processes. In particular, an $\mathbb{R}^n$-valued affine-jump diffusion process X is an $\mathbb{F}$-Markov process whose dynamics is given by

$$dX_t = \delta(t, X_t)dt + \sigma(t, X_t)dW_t + \sum_{h=1}^{m} dJ_t^h, \tag{4}$$

where W is a $\mathbb{F}$-standard Brownian motion in $\mathbb{R}^n$ and each component J^h is a pure-jump process in $\mathbb{R}^n$ with jump-arrival intensity $\left\{\eta^h(t, X_t) : t \geq 0\right\}$ and time-dependent jump distribution ν_t^h on $\mathbb{R}^n$. An important requirement of affine processes is that the drift $\delta : D \to \mathbb{R}^n$, the instantaneous covariance matrix $\sigma\sigma^{\mathrm{T}} : D \to \mathbb{R}^{n \times n}$ and the jump-arrival intensity $\eta^h : D \to \mathbb{R}_+$ must all have an affine dependency on X . The jump-size distribution is determined by its Laplace transform.

Following Schrager (2006), we now assume that the mortality intensity for an individual aged $x+t$ at time t, $\mu_{x+t}(t)$, can be expressed in general form by the following parametric equation

$$\mu_{x+t}(t) = g_0(x, t) + \sum_{j=1}^{M} g_j(x, t) X_j(t), \tag{5}$$

where $g_j : x \to \mathbb{R}_+$ is some real function , possibly dependent on age, and $X_j(t)$ are multiple latent factors conveying mortality dynamics. Contrary to previous formulations, we explicitly assume that jumps have a role in explaining mortality behaviour and assume the multidimensional dynamics of the M latent factors is given by affine-jump processes with diffusion equation

$$dX(t) = \delta(\theta - X(t))\, dt + \Sigma\sqrt{V_t}dW_t^{\mathbb{P}} + dJ_t, \qquad X(0) = \bar{X}, \tag{6}$$

where $W_t^{\mathbb{P}}$ is a $M-$dimensional Brownian motion, J_t denotes a jump component, δ and Σ are $M \times M$ matrices, θ is a vector of dimension M and V_t is a diagonal matrix comprising the diffusion coefficients of the factors on the diagonal. We further assume that the instantaneous drift, the instantaneous covariance matrix and the jump-arrival intensity are affine functions of the latent factors. Contrary to previous formulations, that consider a single age/cohort, equation (5) allows us to model the intensity $\mu_{x+t}(t)$ for all ages simultaneously.

Based on the well know literature on affine term structure models (see, e.g., Duffie e Kan (1996)), we now admit that the survival probability can be represented by an exponentially affine function of the latent factors. Formally, the survival probability of an individual aged $x+t$ in the $(T-t)$ time horizon is given by

$${}_{T-t}p_{x+t}(t) = \mathbb{E}\left[\exp\left(-\int_t^T \mu_{x+s}(s)\, ds\right)\middle|\mathcal{F}_t\right].$$

From the Feynman-Kac theorem, it follows that ${}_{T-t}p_{x+t}(t) \doteq \psi(t, X_t)$ is a solution for the following partial differential equation (simplifying notation from

$X_j(t)$ to X_{tj} and from $\psi(t, X_t)$ to ψ)

$$\frac{d\psi}{dt} + \left(\theta'\delta' - X_t'\delta'\right)\frac{d\psi}{dX_t} + \frac{1}{2}\sum_{k,j=1}^{M}\left(\Sigma V_t \Sigma'\right)_{kj}\frac{d^2\psi}{dX_{tk}dX_{tj}}$$
$$+\sum_{h=1}^{m}\eta^h(X_t,t)\int_{\mathbb{R}^n}\left[\psi(X_t+z,t)-\psi(X_t,t)\right]d\nu_t^h(z) - \left[g_0(x+t)+X_t'g(x+t)\right]\psi = 0.$$

where $g(x+s)$ is a vector

$$g(x+s) = \left[g_1(x+s),\ldots,g_M(x+s)\right]'. \tag{7}$$

Note that in $J_t = \sum_{h=1}^{m} J_t^h$ each jump-type h has a distribution function ν_t^h at time t, dependent only on time t, and a jump-arrival $\left\{\eta^h(t,X_t) : t \geq 0\right\}$ for $h \in \{1,\ldots,m\}$, with η^h defined by $\eta^h(t,X) = \eta_0^h(t) + \eta_1^h(t)\cdot X$. The jump-size distribution ν_t^h of the h^{th} jump process is determined by its Laplace Transform

$$\zeta^h(t,c) = \int_{\mathbb{R}^n} e^{c\cdot z}d\nu_t^h(z) \tag{8}$$

defined in $t\in[0,\infty)$, $c \in \mathbb{C}^n$ and such that the integral is finite.

Le us now assume that the survival probability ${}_{T-t}p_{x+t}(t)$ is represented by the following exponentially affine function

$${}_{T-t}p_{x+t}(t) = \exp\left\{\mathcal{A}(x,t,T) + \mathcal{B}(x,t,T)X_t\right\}. \tag{9}$$

Substituting in the above equation, we get

$$\Bigg[\dot{\mathcal{A}}_t + \dot{\mathcal{B}}_t'X_t + \left(\theta'\delta' - X_t'\delta'\right)\mathcal{B}_t + \frac{1}{2}\sum_{k,j=1}^{M}\sum_{i=1}^{M}\left\{\Sigma_{ki}\left(\alpha_i+\beta_i'X_t\right)\Sigma_{ji}\mathcal{B}_{tk}\mathcal{B}_{tj}\right\} \tag{10}$$
$$+\sum_{h=1}^{m}\left(\eta_0^h+\eta_1^hX_t\right)\left[\zeta^h(t,\mathcal{B}_t)-1\right] - g_0(x+t) - X_t'g(x+t)\Bigg]\psi = 0.$$

By noting that

$$\sum_{k,j=1}^{M}\sum_{i=1}^{M}\left\{\Sigma_{ki}\left(\alpha_i+\beta_i'X_t\right)\Sigma_{ji}\mathcal{B}_{tk}\mathcal{B}_{tj}\right\} = \sum_{i=1}^{M}\left\{\left(\alpha_i+\beta_i'X_t\right)\left(\sum_{k=1}^{M}\Sigma_{ki}\mathcal{B}_{tk}\right)\left(\sum_{j=1}^{M}\Sigma_{ji}\mathcal{B}_{tj}\right)\right\}$$

and that

$$\sum_{k=1}^{M}\Sigma_{ki}\mathcal{B}_{tk} = \left[\Sigma'\mathcal{B}(x,t,T)\right]_i,$$

equation (10) can be simplified to

$$\dot{\mathcal{A}}_t + \dot{\mathcal{B}}_t' X_t + \left(\theta'\delta' - X_t'\delta'\right)\mathcal{B}_t + \frac{1}{2}\sum_{i=1}^{M}\left\{\left[\Sigma'\mathcal{B}_t\right]_i^2\left(\alpha_i + \beta_i' X_t\right)\right\} \tag{11}$$
$$+\sum_{h=1}^{m}\left(\eta_0^h + \eta_1^h X_t\right)\left[\zeta^h\left(t,\mathcal{B}_t\right) - 1\right] - g_0\left(x+t\right) - X_t' g\left(x+t\right) = 0,$$

where coefficients $\mathcal{A}_t \equiv \mathcal{A}(x,t,T)$ and $\mathcal{B}_t \equiv \mathcal{B}(x,t,T)$ are solutions to the following system of Riccati ODE

$$\dot{\mathcal{A}}_t = -\theta'\delta'\mathcal{B}_t - \frac{1}{2}\sum_{i=1}^{M}\left[\Sigma'\mathcal{B}_t\right]_i^2\alpha_i - \sum_{h=1}^{m}\eta_0^h\left[\zeta^h\left(t,\mathcal{B}_t\right) - 1\right] + g_0\left(x+t\right) \tag{12}$$

$$\dot{\mathcal{B}}_t = \delta'\mathcal{B}_t - \frac{1}{2}\sum_{i=1}^{M}\left[\Sigma'\mathcal{B}_t\right]_i^2\beta_i - \sum_{h=1}^{m}\eta_1^h\left[\zeta^h\left(t,\mathcal{B}_t\right) - 1\right] + g\left(x+t\right), \tag{13}$$

where $\dot{\mathcal{A}}_t \equiv \frac{d\mathcal{A}_t}{dt}$ and $\dot{\mathcal{B}}_t \equiv \frac{d\mathcal{B}_t}{dt}$.

The above formulation (5)-(6) of the affine-jump multiple latent factor model is too general for application purposes. In order to adapt it to an actuarial and financial context, latent factors must have a clear interpretation in explaining mortality dynamics over time. For that, we revisit in the next section the classical Gompertz-Makeham mortality law, used normally in a deterministic context to graduate contemporaneous lifetable experiences. We show that for some parametric formulations, closed-form solutions for the survival probability may be derived.

3 Revisiting the classical Gompertz-Makeham law

To illustrate the use of this approach we revisit as Schrager (2006) the classical Gompertz-Makeham mortality law. In its original formulation, the law establishes the following deterministic mathematical relation between age and mortality intensity

$$\mu_{x+t} = X_1 + X_2 c^{x+t} \tag{14}$$

where $X_1 > 0$, $X_2 > 0$ and $c > 1$.

Equation (14) recognises that there may other causes of death other than ageing, an assumption that seems reasonable when we think on the importance of accidental deaths at younger ages. This law can be fitted into the general framework (5) by noting that $g_0(x) = 0$, $g_1(x) = 1$ and $g_2(x) = c^{x+t}$.

Assuming that equation (14) fits the pattern of mortality for all ages appropriately, changes in the intensity $\mu_{x+t}(t)$ can be expressed in terms of variations of the parameters (latent factors) that represent it. In other words, in this model the uncertainty is reflected by the fact that the future paths of the equation parameters are actually unknown.

Choosing a particular functional form for the intensity $\mu_{x+t}(t)$ involves obviously some risk, namely stochastic process risk. However, in this case we can somehow measure and control the risk in a systematic way since we can always select the most appropriate mathematical function, i.e., the one that minimizes the fitting error.

Assume now that all (or at least some) parameters of equation (14) follow stochastic processes as defined by (6). In order to derive analytical solutions for the survival probability, we first confine our analysis to gaussian factor dynamics, i.e., we consider that factor dynamics is driven by multivariate Ornstein-Uhlenbeck processes with jumps. Finally, and without lost of generality, we assume that parameter c is constant over time. The result is the following model

$$\mu_{x+t}(t) = X_1(t) + X_2(t)c^{x+t}, \tag{15}$$

where factors X_j $(j = 1, 2)$ have a dynamic behaviour given by the following SDE

$$\begin{aligned} dX_j(t) &= a_j\left(\theta_j - X_j(t)\right)dt + \sigma_j dW_{jt}^{\mathbb{P}} + dJ_j\left(t\right), \quad X_j(0) = \bar{X}_j \\ dW_{1t}^{\mathbb{P}} dW_{2t}^{\mathbb{P}} &= \rho dt, \end{aligned} \tag{16}$$

where $a_j > 0$, $\theta_j > 0$, $\sigma_j \geq 0$, $W_1^{\mathbb{P}}$ and $W_2^{\mathbb{P}}$ are correlated Brownian movements under the real world probability measure.

We assume that $J(t) = \sum_{i=1}^{N_t} \varepsilon_i$ is a compound Poisson process, independent of W, with constant jump-arrival intensity $\eta \geq 0$, where $\{\varepsilon_i : i = 1, \ldots, \infty\}$ are i.i.d. variables. Following the results by Kou (2002), among others, we consider jump sizes that are random variables double asymmetric exponentially distributed with density

$$f\left(z\right) = \pi_1 \left(\frac{1}{\upsilon_1}\right) e^{-\frac{z}{\upsilon_1}} \mathbb{I}_{\{z \geq 0\}} + \pi_2 \left(\frac{1}{\upsilon_2}\right) e^{\frac{z}{\upsilon_2}} \mathbb{I}_{\{z < 0\}} \tag{17}$$

where $\pi_1, \pi_2 \geq 0$, $\pi_1 + \pi_2 = 1$, represent, respectively, the probabilities of a positive (with average size $\upsilon_1 > 0$) and negative (with average size $\upsilon_2 > 0$) jump. By setting $\pi_1 = 0$ we are interested only on the importance of longevity risk (see, e.g., Biffis, 2005). By setting $\eta = 0$ the model becomes deterministic. When $\upsilon_1 = \upsilon_2$ and $\pi_1 = \pi_2 = \frac{1}{2}$ we get the so-called "first Laplace law". Contrary to other models, by adopting equation (17) we consider the significance of both positive mortality shocks (e.g., new medical breakthroughs) and negative mortality shifts (e.g., bird flu).

Let us now assume that the survival probability ${}_{T-t}p_{x+t}(t)$ is represented by an exponentially affine function, i.e.,

$$\begin{aligned} {}_{T-t}p_{x+t}(t) &= \mathbb{E}\left[\exp\left(-\int_t^T \mu_x(s)ds\right)\Big|\,\mathcal{F}_t\right] = \psi\left(t, X_t\right) \\ &= \exp\left\{\mathcal{A}(x,t,T) + \sum_{j=1}^{2}\mathcal{B}_j(x,t,T)X_j\right\}. \end{aligned} \tag{18}$$

where $\tau = T - t$.

It can be shown that the solution to this problem admits the following Feynman-Kac representation

$$\psi\left(t, X_t\right)\left\{-\dot{\mathcal{A}}(\tau) - \sum_{j=1}^{2}\dot{\mathcal{B}}_j(\tau)X_j + \sum_{j=1}^{2}a_j\left[\theta_j - X_j(t)\right]\mathcal{B}_j(\tau) + \frac{1}{2}\sum_{j=1}^{2}\sigma_j^2\mathcal{B}_j^2(\tau)\right.$$
$$\left. + \mathcal{B}_1(\tau)\mathcal{B}_2(\tau)\sigma_1\sigma_2\rho + \sum_{j=1}^{2}\eta_j\left(\frac{\pi_{1j}}{1 - \upsilon_{1j}\mathcal{B}_j(\tau)} + \frac{\pi_{2j}}{1 + \upsilon_{2j}\mathcal{B}_j(\tau)} - 1\right) - \left(X_1 + X_2c^{x+t}\right)\right\} = 0.$$

Dividing both sides of this equation by $\psi\left(t, X_t\right)$ we get, after some algebra,

$$\begin{aligned} &\left[-\dot{\mathcal{B}}_1(\tau) - a_1\mathcal{B}_1(\tau) - 1\right]X_1 + \left[-\dot{\mathcal{B}}_2(\tau) - a_2\mathcal{B}_2(\tau) - c^{x+t}\right]X_2 - \dot{\mathcal{A}}(\tau) \\ &+ \sum_{j=1}^{2}a_j\theta_j\mathcal{B}_j(\tau) + \sum_{j=1}^{2}\frac{\sigma_j^2}{2}\mathcal{B}_j^2(\tau) + \mathcal{B}_1(\tau)\mathcal{B}_2(\tau)\sigma_1\sigma_2\rho \\ &+ \sum_{j=1}^{2}\eta_j\left(\frac{\pi_{1j}}{1 - \upsilon_{1j}\mathcal{B}_j(\tau)} + \frac{\pi_{2j}}{1 + \upsilon_{2j}\mathcal{B}_j(\tau)} - 1\right) = 0, \end{aligned}$$

where $\mathcal{A}(\tau)$ and $\mathcal{B}_j(\tau)$ $(j = 1, 2)$ are solutions to the following system of Riccati ODE

$$\dot{\mathcal{B}}_1(\tau) = -a_1\mathcal{B}_1(\tau) - 1 \tag{19}$$

$$\dot{\mathcal{B}}_2(\tau) = -a_2\mathcal{B}_2(\tau) - c^{x+t} \tag{20}$$

$$\begin{aligned} \dot{\mathcal{A}}(\tau) &= \sum_{j=1}^{2}a_j\theta_j\mathcal{B}_j(\tau) + \sum_{j=1}^{2}\frac{\sigma_j^2}{2}\mathcal{B}_j^2(\tau) + \mathcal{B}_1(\tau)\mathcal{B}_2(\tau)\sigma_1\sigma_2\rho \\ &\quad \sum_{j=1}^{2}\eta_j\left(\frac{\pi_{1j}}{1 - \upsilon_{1j}\mathcal{B}_j(\tau)} + \frac{\pi_{2j}}{1 + \upsilon_{2j}\mathcal{B}_j(\tau)} - 1\right) \end{aligned} \tag{21}$$

with boundary conditions

$$\mathcal{A}(0)=0, \quad \mathcal{B}_j(0)=0, \quad j=1,2. \tag{22}$$

Admit now, without loss of generality, that $\pi_{12}=0$ (from which $\pi_{22}=1$), i.e., only negative shocks are expected for latent factor X_2.

Solving equation system (19)-(20)-(21) e (22), we finally derive closed-form solutions for $\mathcal{A}(\tau)$ and $\mathcal{B}_j(\tau)$ $(j=1,2)$

$$\mathcal{B}_1(\tau) = \frac{e^{-a_1\tau}-1}{a_1} \tag{23}$$

$$\mathcal{B}_2(\tau) = c^{x+t}\left(\frac{e^{(\xi-a_2)\tau}-1}{a_2-\xi}\right), \quad \xi=\ln c \tag{24}$$

$$\begin{aligned}\mathcal{A}(\tau) = & -\theta_1\left[\mathcal{B}_1(\tau)+\tau\right]+\frac{a_2\theta_2}{\xi-a_2}\left[\tau c^{x+t}-B_2(\tau)\right] \\ & +\frac{\sigma_1^2}{2a_1^2}\left[\tau+\mathcal{B}_1(\tau)\right]-\frac{\sigma_1^2}{4a_1}\mathcal{B}_1^2(\tau) \\ & +\frac{\sigma_2^2 c^{2(x+t)}}{2\left(a_2-\xi\right)^3}\left[\left(a_2-\xi\right)\tau+2e^{(\xi-a_2)\tau}-\frac{1}{2}e^{2(\xi-a_2)\tau}-\frac{3}{2}\right] \\ & +\frac{\rho\sigma_1\sigma_2 c^{x+t}}{a_1\left(a_2-\eta\right)}\left\{\frac{1-e^{(\xi-a_2-a_1)\tau}}{a_1+a_2-\xi}+\mathcal{B}_1(\tau)+\frac{\mathcal{B}_2(\tau)}{c^{x+t}}+\tau\right\} \\ & +\eta_1\left\{\frac{\pi_{11}\left[a_1\tau+\ln\left(1-\upsilon_{11}\mathcal{B}_1(\tau)\right)\right]}{a_1+\upsilon_{11}}+\frac{\pi_{21}\left[a_1\tau+\ln\left(1+\upsilon_{21}\mathcal{B}_1(\tau)\right)\right]}{a_1-\upsilon_{21}}\right. \\ & \left.-\tau\right\}+\eta_2\frac{\left\{\upsilon_{22}c^{x+t}\tau+\ln\left[1+\upsilon_{22}\mathcal{B}_2(\tau)\right]\right\}}{a_2-\xi-\upsilon_{22}c^{x+t}}.\end{aligned} \tag{25}$$

Model (15)-(16) has some advantages over traditional single factor models presented so far. First, if we neglect the importance discontinuous and assume that $\bar{X}_j > \theta_j$, the latent factors exhibit an exponentially decreasing trend, a pattern compatible with decreasing mortality rates observed over time. Second, filling the gap in Schrager (2006) the model captures the four types of mortality risk: random fluctuations (volatility risk), longevity risk (trend risk), catastrophic risk and basis (level) risk. Random fluctuations in mortality are captured by matrix Σ, longevity risk is represented by matrix A, catastrophic mortality risk is captured by the jump component in (16). Finally, basis (adverse selection) risk can be captured by noting that the model is compatible with a relational approach of the type suggested by Brass (1971), linking the mortality experience of the general population with that of life insured, for example.

Third, the model resumes the dynamics of mortality for all age groups (cohorts) throughout human lifespan in a single equation, considering multiple causes of death and their possible correlations. Finally, and despite its analytical complexity, the model admits closed-form solutions for the survival probability making it suitable for estimation and pricing applications within the insurance industry. Recall, however, that by considering gaussian dynamics for the latent factors we do not run out the possibility of negative mortality rates, a feature well know within the interest rate literature. As an alternative, non-gaussian specifications for the factor dynamics could involve, for instance, the use of the Feller equation with jumps

$$\begin{aligned} dX_j(t) &= a_j\left(\theta_j - X_j(t)\right)dt + \sigma_j\sqrt{X_j(t)}dW_{jt}^{\mathbb{P}} + dJ_j\left(t\right),\ X_j(0) = \bar{X}_j \qquad (26) \\ dW_{1t}^{\mathbb{P}}dW_{2t}^{\mathbb{P}} &= \rho dt. \end{aligned}$$

It can be shown that in this case the solution to this problem admits the following Feynman-Kac representation

$$\begin{aligned} &\psi\left(t, X_t\right)\Bigg\{-\dot{\mathcal{A}}(\tau) - \sum_{j=1}^{2}\dot{\mathcal{B}}_j(\tau)X_j + \sum_{j=1}^{2}a_j\left[\theta_j - X_j(t)\right]\mathcal{B}_j(\tau) + \frac{1}{2}\sum_{j=1}^{2}\sigma_j^2\mathcal{B}_j^2(\tau)X_j \\ &+\mathcal{B}_1(\tau)X_1\mathcal{B}_2(\tau)X_2\sigma_1\sigma_2\rho + \sum_{j=1}^{2}\eta_j\left(\frac{\pi_{1j}}{1-\upsilon_{1j}\mathcal{B}_j(\tau)} + \frac{\pi_{2j}}{1+\upsilon_{2j}\mathcal{B}_j(\tau)} - 1\right) \\ &-(X_1 + X_2c^{x+t})\Bigg\} = 0. \end{aligned}$$

Up until now, there is a lack of empirical evidence supporting the correlation between Gompertz-Makeham Wiener processes and so, without loss of generality, we assume that $\rho = 0$.

Dividing both sides of this equation by $\psi\left(t, X_t\right)$ we get

$$\begin{aligned} &\left[-\dot{\mathcal{B}}_1(\tau) - a_1\mathcal{B}_1(\tau) + \frac{\sigma_1^2}{2}\mathcal{B}_1^2(\tau) - 1\right]X_1 + \left[-\dot{\mathcal{B}}_2(\tau) - a_2\mathcal{B}_2(\tau) + \frac{\sigma_2^2}{2}\mathcal{B}_2^2(\tau) - c^{x+t}\right]X_2 \\ &-\dot{\mathcal{A}}(\tau) + \sum_{j=1}^{2}a_j\theta_j\mathcal{B}_j(\tau) + \sum_{j=1}^{2}\eta_j\left(\frac{\pi_{1j}}{1-\upsilon_{1j}\mathcal{B}_j(\tau)} + \frac{\pi_{2j}}{1+\upsilon_{2j}\mathcal{B}_j(\tau)} - 1\right) = 0, \end{aligned}$$

where $\mathcal{A}(\tau)$ and $\mathcal{B}_j(\tau)$ $(j = 1, 2)$ are once again solutions of the following system of ODE

$$\begin{aligned}
\dot{\mathcal{B}}_1(\tau) &= -a_1\mathcal{B}_1(\tau) + \frac{1}{2}\sigma_1^2\mathcal{B}_1^2(\tau) - 1 && (27)\\
\dot{\mathcal{B}}_2(\tau) &= -a_2\mathcal{B}_2(\tau) + \frac{1}{2}\sigma_2^2\mathcal{B}_2^2(\tau) - c^{x+t} && (28)\\
\dot{\mathcal{A}}(\tau) &= \sum_{j=1}^{2} a_j\theta_j\mathcal{B}_j(\tau) + \sum_{j=1}^{2}\eta_j\left(\frac{\pi_{1j}}{1-\upsilon_{1j}\mathcal{B}_j(\tau)} + \frac{\pi_{2j}}{1+\upsilon_{2j}\mathcal{B}_j(\tau)} - 1\right) && (29)
\end{aligned}$$

with boundary conditions

$$\mathcal{A}(0) = 0, \quad \mathcal{B}_j(0) = 0, \quad j = 1, 2. \tag{30}$$

By solving (27)-(28)-(29) e (30), we get a closed-form solution for $\mathcal{B}_1(\tau)$

$$\mathcal{B}_1(\tau) = \frac{1-e^{-\kappa_1\tau}}{\phi_1 + \xi_1 e^{-\kappa_1\tau}}, \qquad \text{com} \quad \begin{cases} \kappa_1 = \sqrt{a_1^2 + 2\sigma_1^2} \\ \phi_1 = -\frac{(a_1+\kappa_1)}{2} \\ \xi_1 = \frac{(a_1-\kappa_1)}{2} \end{cases} \tag{31}$$

while $\mathcal{B}_2(x,t,T)$ and $\mathcal{A}(x,t,T)$ can only be solved by numerical methods.

4 Conclusion

In this paper we assume that the mortality intensity can be described by an affine function of a finite number of latent factors whose dynamics is represented by affine-jump diffusion processes. We explicitly assume that jumps have a role in explaining mortality behaviour. Specifically, we consider jump sizes that are random variables double asymmetric exponentially distributed. The model is compatible with both negative and positive jumps in mortality, a feature that contrasts with similar models that are interested in sudden improvements in mortality (e.g., due to medical advances) only. Model application is illustrated revisiting the classical Gompertz-Makeham mortality law considering both gaussian and non-gaussian factor dynamics. In the former case, survival probabilities have been provided in closed-form.

References

[1]Artzner, P. and Delbaen, F. (1995). Default risk insurance and incomplete markets. Mathematical Finance, 5(3), 187-195.

[2]Biffis, E. (2005). Affine processes for dynamic mortality and actuarial valuations. Insurance: Mathematics and Economics, 37, 443-468.

[3]Biffis, E. and Denuit, M. (2005). Lee Carter Goes Risk-Neutral: An Application to the Italian Annuity Market. Actuarial Research Paper N.^{o} 166, Faculty of Actuarial Science and Statistics, Cass Business School, City University, London.

[4]Biffis, E. and Millossovich, P. (2004). A Bidimensional Approach to Mortality Risk. Working Paper, Università Bocconi.

[5]Biffis, E., Denuit, M. and Devolder, P. (2006). Stochastic mortality under measure changes. Actuarial Research Paper, Faculty of Actuarial Science and Statistics, Cass Business School, City University, London.

[6]Booth, H. (2006). Demographic forecasting: 1980 to 2005 in review. *International Journal of Forecasting*, 22 (3), 547-581.

[7]Brass, W. (1971). On the scale of mortality. In: *Biological Aspects of Demography*, London Taylor and Francis.

[8]Bravo, J. M. (2007). Tábuas de Mortalidade Contemporâneas e Prospectivas: Modelos Estocásticos, Aplicações Actuariais e Cobertura do Risco de Longevidade. PhD Thesis, University of Évora, Portugal.

[9]Dahl, M. (2004). Stochastic Mortality in Life Insurance: Market Reserves and Mortality-Linked Insurance Contracts. Insurance: Mathematics and Economics, 35, 113-136.

[10]Dahl, M. and Moller, T. (2005). Valuation and hedging of life insurance liabilities with systematic mortality risk. In Proceedings of the 15th International AFIR Colloquium, Zurich.

[11]GAD – Government Actuary's Department (2001). National Population Projections: Review of Methodology for Projecting Mortality. Report N.^{o} 8, London.

[12]Kou, S. (2002). A jump-diffusion model for option pricing. Management Science, 48, 1086-1101.

[13]Lee, R. and Carter, L. (1992). modelling and forecasting the time series of US mortality. Journal of the American Statistical Association, 87, 659-671.

[14]Milevsky, M. and Promislow, S. (2001). Mortality derivatives and the option to annuitise. Insurance: Mathematics and Economics, 29, 299-318.

[15]Pitacco, E. (2004). Survival models in a dynamic context: a survey. Insurance: Mathematics and Economics, 35, 279-298.

[16]Schrager, D. (2006). Affine Stochastic Mortality. Insurance: Mathematics and Economics, 38, 81-97.

[17]Tabeau, E. (2001). A review of demographic forecasting models for mortality. In E. Tabeau et al. (Eds), Forecasting Mortality in Developed Countries: insights from a statistical, demographical and epidemiological perspective, Kluwer Academic Publishers, 1-32.

[18]Tuljapurkar, S. and Boe, C. (1998). Mortality change and forecasting: how much and how little do we know?. North American Actuarial Journal, 2(4), 13-47.

[19]Wong-Fupuy, C. and Haberman, S. (2004). Projecting mortality trends: recent developments in the United Kingdom and the United States. North American Actuarial Journal, 8(2), 56-83.