What's gone wrong in the design of PAYG systems?

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March, 2019

Abstract

In order to face the population ageing problem, most countries with PAYG systems introduced pension reforms during the last twenty years. However, in some cases these reforms are considered as insufficient to guarantee the pension sustainability; in other cases, the pension sustainability is achieved through the introduction of drastic reforms and, thus, at the expense of a dramatic reduction in the well-being of current and future generations. The objective of this article is to show that the non-sustainability of PAYG systems and, consequently, the necessity to introduce drastic pension reforms, is explained by the fact that in countries with PAYG systems pensions have not been computed according to appropriate rules. In particular, we show that the sustainability of the pension system is guaranteed if (i) pension benefits are computed using actuarial principles, (ii) the implicit rate of return on contributions is the same for each retiree and equal to the average wage bill growth rate, and (iii) pension reserves are remunerated at a risk-free interest rate equal to the average wage bill growth rate. These conditions allow a PAYG system to face any demographic shock, such as an increase in life expectancy and a transitory increase in fertility rates (baby boom) followed by a transitory reduction in fertility rates (baby boost).

Keywords: Pension economics; Pension finance; Population ageing.

JEL classification: H55; H6.

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1 Introduction

Most pension systems, especially in Europe, are public PAYG systems (where the pensions perceived by current retirees are financed using the contributions paid by current workers) with earning-related pension benefits. With respect to fully-funded systems, which are based on the accumulation of financial assets, PAYG systems are based on the promise that people who pay contributions when active will receive pension benefits when retired. According to Barr (2002), any public pension system should have the following goals: to provide a mechanism for consumption smoothing, to provide income security for the elderly and to reduce poverty of the elderly. To these objectives, it is important to add the principle that the pension system has to guarantee equal treatment to all individuals belonging to the same generation and to all generations (see, for instance, Oksanen, 2009). However, these principles are quite general which implies the necessity to define the weights to attribute to each of these objectives and, in a broader sense, to define the role of the State in the field of pensions. More concretely, it is necessary to define precisely the computation rules of pension benefits in order to achieve the objectives outlined above.

It is well-known that PAYG pension systems are vulnerable to demographic shocks.[1] In order to face the population ageing problem, induced by a simultaneous increase in life expectancy and reduction in fertility rates, several countries introduced pension reforms during the last twenty years. However, in some cases, these reforms are considered as insufficient to guarantee the sustainability of PAYG pension systems; in other cases, the sustainability is achieved through the introduction of drastic reforms and, thus, at the expense of current and future generations that incur a considerable reduction in their well-being. Some countries like Sweden, Italy, Poland and Latvia have introduced structural reforms, namely the introduction of an NDC (Notional Defined Contribution) system[2] which permits to better link contributions paid and pensions earned and, consequently, to reduce the distortions in the labor market. However, if population is ageing, the introduction of an NDC system is not sufficient to solve the sustainability problem of pension systems and additional reforms are needed. Settergren and Mikula (2005), Robalino and Bodor (2009) and Knell (2016) propose sophisticated adjustment mechanisms of pension benefits that guarantee the sustainability of NDC pension systems. In any case, the sustainability of

[1] However, as noted by Barr (2002), even fully-funded schemes are exposed to negative demographic shocks because they induce a fall in the value of financial assets and then in the value of pensions.

[2] For a description of NDC systems, see Valdés-Prieto (2000), Disney (1999), and Börsch-Supan (2005).

the pension system can be achieved only through a reduction in the implicit rate of return on contributions for current and future generations which can be seen as a break of past promises.

The objective of this article is not to identify the reforms aiming to ensure the sustainability of PAYG systems. If a pension system is not sustainable, it is very simple, at least from a technical point of view, to make it sustainable and the well-known solutions are (i) the increase in the contribution rate, (ii) the increase in the retirement age and (iii) the reduction in the level of pension benefits.[3] In the first two cases, the burden of the reform is borne by workers or by firms, while in the second case it is borne by retirees. All these cases have important drawbacks. In particular, an increase in contributions may reduce individuals' labor participation or firms' competitiveness, while a reduction in pensions may be considered as a break of past promises and may compromise the achievement of the objective of poverty relief (Barr, 2002). Furthermore, the increase in the retirement age or in the contribution rate can be interpreted as a break of past promises if the implicit rate of return on contributions falls for the individuals concerned by these reforms. Thus, each policy reform has different implications on intergenerational equity and efficiency. The issue of generational inequities generated by pension reforms has been studied, among others, by Auerbach and Lee (2011) who compare different PAYG systems and analyze how the risks associated with demographic shocks are spread among generations, and by Beetsma and Oksanen (2007) who focus on European PAYG systems and analyze the consequences on the intergenerational distribution of different fiscal policies. Concerning the issue of efficiency, Flodén (2003) shows that for countries that are characterized by a large public sector, the optimal policy is to smooth the tax rate over time in order to avoid excessively high tax adjustments in the future and thus important distortions in the labor market. In contrast, Andersen (2012) argues that tax smoothing is not optimal when population ageing is driven by an increase in longevity since, in this case, the optimal policy is to increase the retirement age.

The objective of this article is to investigate the reasons of the non-sustainability of PAYG systems. We show that the main reason is that, in the past, pension benefits were not computed according to appropriate rules. In particular, we argue in this article that if pensions are computed using actuarial principles and such that, for all individuals, the implicit rate of return on contributions is equal to the *average* wage bill growth rate, then a PAYG system is able to

[3]The financial situation of the pension system may also be improved by introducing labor market policies aiming at raising women's labor participation or immigration policies; however, the positive effect of these reforms is only transitory since, sooner or later, the new workers will earn pension benefits; thus, these policies cannot solve the pension sustainability problem.

face any demographic shock. In the case of an increase in life expectancy, this rule guarantees that the pension system is sustainable if the increase in survival probabilities is (perfectly or at least partially) anticipated and taken into account in the computation of pension benefits. In the case of a transitory increase in fertility rates (baby boom) followed by a transitory reduction in fertility rates (baby boost), this rule implies that the pension system accumulates reserves during the baby boom that can be used when population is ageing. In particular, we show that the pension system is sustainable if pension reserves are remunerated at a risk-free interest rate equal to the average wage bill growth rate. The sustainability problem is thus explained by the fact that (i) the increase in life expectancy has not been correctly anticipated and taken into account in the computation of pensions; (ii) during the baby boom, where the number of retirees was considerably low with respect to the number of workers, countries with PAYG systems had not accumulated sufficient pension reserves to face the future population ageing problem. In other words, instead of building adequate pension reserves, the high amount of contributions has been used to pay pensions implying an excessive generosity to retirees during the baby boom. In particular, these generations obtained very high (and unsustainable) implicit rates of return on contributions which can be interpreted as a gift that, sooner or later, must be paid by future generations in order to make the system sustainable.

Interestingly, the role of pension reserves has been underestimated in the literature. For instance, Barr and Diamond (2009) state that "The state can but does not have to accumulate assets in anticipation of future pension claims"; Robalino and Bodor (2009) state that "Many of the systems can have reserves, but these act rather as a "buffer stock" to smooth adjustments to benefits or contribution rates that become necessary as a result of unexpected macroeconomic and/or demographic shocks or the gradual maturation of the system"; Oksanen (2001) states that "Given that in most cases pension reforms will not be sufficient to guarantee pension sustainability, there is a need for public PAYG systems to build reserves." Thus, the creation of pension reserves has been considered as an option (but not as a necessary condition to guarantee the sustainability and the fairness of the pension system) and as a useful policy that can be implemented if the system is unsustainable in order to avoid excessive future adjustments. Even though some countries have established public pension reserves, only in Japan, Korea, and Sweden public pension reserves represent more than 20% of GDP in 2008 (see table 1).

The numerical simulations presented in the second part of the article, which analyze the effects of a realistic demographic shock, confirm that a PAYG system is sustainable if the demographic shock is perfectly anticipated or at least partially anticipated. Clearly, one can argue that this assumption is unrealistic. However, the existence of a population ageing problem was known at least in 1974, as documented by the World Population Conference organized by the United Nations in Bucharest in August 1974.[4] Even though, probably, the magnitude of the demographic shock was not perfectly known in 1974, the fall in mortality and in fertility rates was already predicted. The simulation analysis based on more reasonable assumptions (where the demographic shock is assumed to be unanticipated until 1980) confirms that a PAYG system can face the demographic shock if pension benefits are computed according to actuarial principles.

The article is organized as follows. Section 2 presents the theoretical model and the analysis of different demographic shocks on the sustainability of a PAYG pension system, which is assumed to be an NDC system.[5] In particular, we analyze the effects of a temporary and "symmetric" shock on fertility rates and the effects of a shock on survival probabilities by distinguishing whether this shock is perfectly anticipated, unanticipated, or partially anticipated. Section 3 presents numerical simulations using a stylized OLG model (Auerbach and Kotlikoff, 1987) calibrated on German demographic data. These stylized simulations allow to give an idea of the size of the pension reserves that should have been created in the past decades to face the population ageing problem in the future decades. Section 4 concludes.

2 The OLG model with an NDC pension system

The theoretical model used in this analysis is a simple OLG model with one representative firm, four overlapping generations and an NDC pension system. In particular, we consider a small-open economy in which the representative firm produces one good using a standard Cobb-Douglas technology, $Y_t = K_t^{\alpha} \cdot (A_t \cdot L_t)^{1-\alpha}$, where Y_t indicates real GDP, K_t the stock of capital available in the economy, L_t the number of workers which are assumed to have the same productivity A_t. First order conditions for profit maximization require that $w_t = (1-\alpha) \cdot A_t \cdot [K_t/(A_t \cdot L_t)]^{\alpha}$ and $r_t + \delta = \alpha \cdot [K_t/(A_t \cdot L_t)]^{\alpha-1}$, where w_t is the real wage, r_t is

[4] As indicated at page 8 of the report of the conference (United Nations, 1975): "The World Population Conference recommends that great importance should be attached to the phenomenon of aging. Owing to the decline in fertility and possible medical progress against cancer and the other diseases of old age, aging will sooner or later affect all nations more or less intensely. It will be important to study carefully the social and economic consequences and the repercussions on morale, particularly with regard to a sense of vitality and progress."

[5] We focus on the Notional Account system for convenience. This makes it easier to compute the value of pension benefits if a shock occurs, in particular in the case of a change in survival probabilities.

the interest rate and δ is the depreciation rate. The small-open-economy assumption implies that the interest rate is exogenously fixed at the world level, $r_t = \overline{r}$. Consequently, capital flows adjust in order to keep constant the capital per unit of effective labor, $K_t/(A_t \cdot L_t) = \overline{k}$. Productivity is assumed to grow at a constant and exogenous rate g, implying that the real wage w_t increases at the same rate g.

In this economy, we assume that 4 generations coexist in each period. Thus, people can be classified according to the age group $g1$, $g2$, $g3$ and $g4$ and each period consists of 20 years. This simple framework is sufficiently general and allows to consider in each period more than one generation that work and pay contributions and more than one generation that earns pension benefits. In particular, we assume that all individuals belonging to the first two age groups work and pay contributions according to a constant and exogenous rate τ. All workers, independently of their age, are assumed to have the same productivity A_t and, thus, earn the same wage w_t. All individuals belonging to the last two age groups do not work and earn pension benefits according to the rules described in section 2.1.

We assume that all individuals (except those belonging to the last age group) will be alive in the next period according to an exogenous survival probability ω and that the number of individuals belonging to the first age group increases over time according to an exogenous fertility rate n.[6] The demographic evolution in the model is described by the following equations:

$$N_{g1,t+1} = N_{g1,t} \cdot (1 + n_{t+1}) \tag{1}$$

$$N_{g2,t+1} = N_{g1,t} \cdot \omega_{g2,t+1} = N_{g1,t} \cdot \Omega_{g2,t+1} \tag{2}$$

$$N_{g3,t+1} = N_{g2,t} \cdot \omega_{g3,t+1} = N_{g1,t-1} \cdot \Omega_{g3,t+1} \tag{3}$$

$$N_{g4,t+1} = N_{g3,t} \cdot \omega_{g4,t+1} = N_{g1,t-2} \cdot \Omega_{g4,t+1} \tag{4}$$

where $\omega_{g2,t+1}$ represents the conditional probability for individuals belonging to the first age group in t to be alive in the next period; $\omega_{g3,t+1}$ represents the conditional probability for individuals that belong to the second age group in t to be alive in the next period; $\omega_{g4,t+1}$ is the conditional probability for individuals that belong to the third age group in t to be alive in the next period. Starting from the conditional survival probabilities, it is possible to determine the unconditional survival probabilities: $\Omega_{g2,t+1} = \omega_{g2,t+1}$ is the probability to be alive in $t+1$ and to belong to the second class group; $\Omega_{g3,t+1} = \omega_{g2,t} \cdot \omega_{g3,t+1}$ is the probability to be alive in

[6] Of course, the parameter n represents the growth rate of the number of young people belonging to the first age class. This parameter can be interpreted as a proxy of the fertility rate.

$t+1$ and to belong to the third class group; $\Omega_{g4,t+1} = \omega_{g2,t-1} \cdot \omega_{g3,t} \cdot \omega_{g4,t+1}$ is the probability to be alive in $t+1$ and to belong to the fourth class group.

In the absence of demographic shocks, i.e. if the fertility rates and the survival probabilities remain constant, the population is stable. Thus, the old-age dependency ratio and the population structure remain constant over time.

2.1 The computation of pension benefits

We assume the existence of an NDC pension system. For each individual, pension benefits are computed according to actuarial principles such that the expected present value of pension benefits is equal to the expected capitalized value of contributions.[7] In particular, for an individual who starts working in $t-2$ and earns a pension $P_{g3,t}$ when aged $g3$ in t and a pension $P_{g4,t+1}$ when aged $g4$ in $t+1$, the rule is the following:

$$\tau \cdot \left[w_{t-2} \cdot (1+R)^2 \ + \ w_{t-1} \cdot (1+R) \cdot \Omega_{g2,t-1} \right] \ = \ P_{g3,t} \cdot \Omega_{g3,t} \ + \ \frac{P_{g4,t+1}}{1+R} \cdot \Omega_{g4,t+1} \qquad (5)$$

where R is the notional rate of interest used to compute the first pension (by capitalizing past contributions and by discounting future pensions). The previous pension rule implies that, in the absence of demographic shocks, the implicit rate of return on contributions IRR coincides with the notional rate of interest R. In the case of a change in $\Omega_{g4,t+1}$, the implicit rate of return on contributions IRR coincides with the notional rate of interest R only if the future value of $\Omega_{g4,t+1}$ is perfectly anticipated and taken into account in the computation of pension benefits in t. In contrast, if the increase in $\Omega_{g4,t+1}$ is not anticipated in t, the implicit rate of return on contributions IRR obtained by the individuals who retire in t is higher than the notional rate of interest R.

In this article, we assume that the notional rate of interest is constant, even if the economy is hit by demographic or technological shocks. More precisely, the notional rate of interest is computed as follows:

$$1+R \ = \ (1+\overline{n}) \cdot (1+\overline{g}) \qquad (6)$$

[7] Note that this rule is not standard. In general, in NDC systems, pension benefits are computed considering the capitalized value of contributions, which are not weighted by the survival probabilities. However, in this case, as indicated by Robalino and Bodor (2009), there not exists a closed form solution for the computation of the implicit rate of return on contributions (IRR) that guarantees the sustainability of the pension system (which depends on the wage bill growth rate, but also on survival probabilities and retirement patterns). In contrast, the rule used in my article allows to avoid this problem. As shown later, the IRR compatible with the sustainability of the pension system for an economy that is on its balanced growth path is equal to the wage bill growth rate according to the Aaron-Samuelson condition.

where $\overline{n}$ represents the average (computed over the entire period) fertility rate and $\overline{g}$ represents the average (computed over the entire period) productivity growth rate. Thus, in the absence of shocks or if the shocks are anticipated and taken into account in the computation of pension benefits, all generations receive exactly the same treatment since the implicit rate of return on contributions is the same for all retirees and is equal to the average wage bill growth rate.

Finally, we assume that pensions are indexed on inflation (implying that pensions remain constant in real terms, i.e. $P_{g4,t+1} = P_{g3,t}$). Thus, the first pension in t is given by:

$$\begin{aligned} P_{g3,t} &= \frac{\tau \cdot \left[w_{t-2} \cdot (1+R)^2 \;+\; w_{t-1} \cdot (1+R) \cdot \Omega_{g2,t-1}\right]}{\Omega_{g3,t} + \frac{\Omega_{g4,t+1}}{1+R}} \\ &= w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\Omega_{g2,t-1})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \Omega_{g3,t} + \Omega_{g4,t+1}} \end{aligned} \tag{7}$$

Equation 7 implies that the replacement ratio positively depends on the average fertility rate and the average productivity growth rate and negatively on the probability to be alive at age $g3$ and on the expected probability to be alive at age $g4$.

2.2 The sustainability of the pension system in an economy that is on its balanced growth path

A pension system can be defined as sustainable if the current and future resources are sufficient to meet its commitments. More precisely, following Robalino and Bodor (2009), the gross implicit pension debt (i.e. the present value of the pensions that will be paid to current retirees and to current contributors) must be covered by the current value of pension reserves and the pension asset (i.e. the difference between the present value of future contributions and the present value of pensions ensuing from these contributions). Thus, the pension system is sustainable if the present value of long-run pension reserves is non-negative. This implies that pension reserves, in the long run, must be positive or nil. They can also be negative but, in this case, the pension system is sustainable if the growth rate of the pension debt is lower than the interest rate.

Proposition 1. In the absence of shocks, implying that $n_t = \overline{n}$, $g_t = \overline{g}$, $\Omega_{g2,t} = \overline{\Omega}_{g2}$, $\Omega_{g3,t} = \overline{\Omega}_{g3}$ and $\Omega_{g4,t} = \overline{\Omega}_{g4}$, if pensions are computed according to equation 7, then the pension system is always balanced and, thus, sustainable.

Proof.

The pension surplus in time $t-1$ is given by:

$$S_{t-1} = \tau \cdot w_{t-1} \cdot (N_{g1,t-1} + N_{g2,t-1}) - (P_{g3,t-1} \cdot N_{g3,t-1} + P_{g4,t-1} \cdot N_{g4,t-1}) \tag{8}$$

where:

$$\begin{aligned}
N_{g2,t-1} &= N_{g1,t-2} \cdot \overline{\Omega}_{g2} = \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g2} \\
N_{g3,t-1} &= N_{g1,t-3} \cdot \overline{\Omega}_{g3} = \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g3} \\
N_{g4,t-1} &= N_{g1,t-4} \cdot \overline{\Omega}_{g4} = \frac{N_{g1,t-1}}{(1+\overline{n})^3} \cdot \overline{\Omega}_{g4}
\end{aligned}$$

and:

$$\begin{aligned}
P_{g3,t-1} &= w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
P_{g4,t-1} &= P_{g3,t-2} = w_{t-3} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}
\end{aligned} \tag{9}$$

Thus, the pension surplus in time $t-1$ is given by:

$$\begin{aligned}
S_{t-1} &= \tau \cdot w_{t-1} \cdot \left(N_{g1,t-1} + \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g2} \right) \\
&- w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g3} \\
&- w_{t-3} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^3} \cdot \overline{\Omega}_{g4}
\end{aligned} \tag{10}$$

After some mathematical manipulations,[8] it is possible to find:

$$S_{t-1} = 0 \tag{11}$$

Q.E.D.

Thus, in the absence of shocks, the economy is on its balanced growth path since the population is stable and the pension system is always balanced and thus sustainable. In addition, for all retirees, the implicit rate of return on contributions is equal to the wage bill growth rate, which is consistent with the Aaron-Samuelson condition (Aaron, 1966 and Samuelson, 1958).

[8] For the mathematical details, see Appendix 1a.

2.3 Demographic shock on fertility rates

Here, we analyze the effects of a temporary and symmetric shock on fertility rates. Temporary, in the sense that, after k periods, the fertility rates come back to the initial pre-shock level $\overline{n}$. Symmetric, in the sense that the average fertility rate for the entire period is equal to $\overline{n}$. This means that if the fertility rate increases for some periods, then it must decrease for some other periods such that $\prod_{t=1}^{k}(1+n_t) = (1+\overline{n})^k$. Thus, $\overline{n}$ is the average fertility rate over the k periods.

Proposition 2. In the case of a temporary and symmetric demographic shock on fertility rates, if pension benefits are computed according to equation 7 and if pension reserves are remunerated at the constant and exogenous rate R as defined in equation 6, then a PAYG pension system is sustainable.

Proof.

Without loss of generality, we assume that the fertility rate increases in t (i.e. $n_t > \overline{n}$) and decreases in $t+1$ (i.e. $n_{t+1} < \overline{n}$). Starting from $t+2$, the fertility rate comes back to the initial level $\overline{n}$. In addition, the demographic shock is assumed to be symmetric, implying that $(1+n_t)\cdot(1+n_{t+1}) = (1+\overline{n})^2$.

Until time $t-1$, the population is stable and, given that pensions are computed according to equation 7, the pension system is always balanced as proved in Proposition 1. In particular, before the demographic shock, the old-age dependency ratio is constant. In $t-1$, the old-age dependency ratio, DR, is:

$$DR_{t-1} = \frac{N_{g3,t-1}+N_{g4,t-1}}{N_{g1,t-1}+N_{g2,t-1}} = \frac{\frac{N_{g1,t-1}}{(1+\overline{n})^2}\cdot\overline{\Omega}_{g3}+\frac{N_{g1,t-1}}{(1+\overline{n})^3}\cdot\overline{\Omega}_{g4}}{N_{g1,t-1}+\frac{N_{g1,t-1}}{1+\overline{n}}\cdot\overline{\Omega}_{g2}} = \frac{(1+\overline{n})\cdot\overline{\Omega}_{g3}+\overline{\Omega}_{g4}}{(1+\overline{n})^2\cdot(1+\overline{n}+\overline{\Omega}_{g2})}$$

In time t, the pension surplus is given by:

$$S_t = \tau\cdot w_t\cdot(N_{g1,t}+N_{g2,t}) - (P_{g3,t}\cdot N_{g3,t}+P_{g4,t}\cdot N_{g4,t}) \tag{12}$$

where:

$$
\begin{aligned}
N_{g1,t} &= N_{g1,t-1} \cdot (1+n_t) \\
N_{g2,t} &= N_{g1,t-1} \cdot \overline{\Omega}_{g2} \\
N_{g3,t} &= N_{g1,t-2} \cdot \overline{\Omega}_{g3} = \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g3} \\
N_{g4,t} &= N_{g1,t-3} \cdot \overline{\Omega}_{g4} = \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4}
\end{aligned}
$$

and:

$$
\begin{aligned}
P_{g3,t} &= w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
P_{g4,t} &= P_{g3,t-1} = w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}
\end{aligned}
$$

In time t, the old-age dependency ratio is equal to:

$$
DR_t = \frac{N_{g3,t} + N_{g4,t}}{N_{g1,t} + N_{g2,t}} = \frac{\frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g3} + \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4}}{N_{g1,t-1} \cdot (1+n_t) + N_{g1,t-1} \cdot \overline{\Omega}_{g2}} = \frac{(1+\overline{n}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}{(1+\overline{n})^2 \cdot (1+n_t+\overline{\Omega}_{g2})}
$$

which is lower than the level before the demographic shock because of the increase in the fertility rate in time t.

The pension surplus in time t is given by:[9]

$$
S_t = \tau \cdot w_t \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) > 0 \tag{13}
$$

The previous result implies that, in time t when the economy is characterized by a baby boom and by a reduction in the old-age dependency ratio, the pension system has a positive surplus. This is explained by the fact that contributions paid by the additional workers is not used to pay pensions. Note that the notional rate of interest is still equal to the *average* wage bill growth rate, even if, in time t, the wage bill growth rate increases because of the increase in the fertility rate.

In time $t+1$, the pension surplus is given by:

$$
S_{t+1} = \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) - (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \tag{14}
$$

[9] For the mathematical details, see Appendix 1b.

where:

$$
\begin{aligned}
N_{g1,t+1} &= N_{g1,t} \cdot (1+n_{t+1}) = N_{g1,t-1} \cdot (1+n_t) \cdot (1+n_{t+1}) = N_{g1,t-1} \cdot (1+\overline{n})^2 \\
N_{g2,t+1} &= N_{g1,t} \cdot \overline{\Omega}_{g2} = N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g2} \\
N_{g3,t+1} &= N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\
N_{g4,t+1} &= N_{g1,t-2} \cdot \overline{\Omega}_{g4} = \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4}
\end{aligned}
$$

and:

$$
\begin{aligned}
P_{g3,t+1} &= w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
P_{g4,t+1} &= P_{g3,t} = w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}
\end{aligned}
$$

In time $t+1$, the old-age dependency ratio is equal to:

$$
\begin{aligned}
DR_{t+1} &= \frac{N_{g3,t+1} + N_{g4,t+1}}{N_{g1,t+1} + N_{g2,t+1}} = \frac{N_{g1,t-1} \cdot \overline{\Omega}_{g3} + \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4}}{N_{g1,t-1} \cdot (1+\overline{n})^2 + N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g2}} \\
&= \frac{(1+\overline{n}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}{(1+\overline{n})^2 \cdot \left(1+\overline{n}+\frac{1+n_t}{1+\overline{n}} \cdot \overline{\Omega}_{g2}\right)}
\end{aligned}
$$

which is again lower than the level observed before the demographic shock because the size of the age class $g2$ is higher than before the demographic shock, while the size of all other age classes (including the first one since the shock is assumed to be symmetric) is the same.

The pension surplus in time $t+1$ is given by:[10]

$$
S_{t+1} = \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) \cdot \overline{\Omega}_{g2} > 0 \tag{15}
$$

Thus, in time $t+1$, the pension system still has a positive surplus. This is explained by the fact that the old-age dependency ratio is still lower than the initial level.

In time $t+2$, the pension surplus is given by:

$$
S_{t+2} = \tau \cdot w_{t+2} \cdot (N_{g1,t+2} + N_{g2,t+2}) - (P_{g3,t+2} \cdot N_{g3,t+2} + P_{g4,t+2} \cdot N_{g4,t+2}) \tag{16}
$$

[10] For the mathematical details, see Appendix 1c.

where:

$$
\begin{aligned}
N_{g1,t+2} &= N_{g1,t+1} \cdot (1+\overline{n}) &= N_{g1,t-1} \cdot (1+\overline{n})^3 \\
N_{g2,t+2} &= N_{g1,t+1} \cdot \overline{\Omega}_{g2} &= N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot \overline{\Omega}_{g2} \\
N_{g3,t+2} &= N_{g1,t} \cdot \overline{\Omega}_{g3} &= N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g3} \\
N_{g4,t+2} &= N_{g1,t-1} \cdot \overline{\Omega}_{g4}
\end{aligned}
$$

and:

$$
\begin{aligned}
P_{g3,t+2} &= w_{t+1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
P_{g4,t+2} &= P_{g3,t+1} = w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}
\end{aligned}
$$

In time $t+2$, the old-age dependency ratio is equal to:

$$
\begin{aligned}
DR_{t+2} &= \frac{N_{g3,t+2} + N_{g4,t+2}}{N_{g1,t+2} + N_{g2,t+2}} = \frac{N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g3} + N_{g1,t-1} \cdot \overline{\Omega}_{g4}}{N_{g1,t-1} \cdot (1+\overline{n})^3 + N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot \overline{\Omega}_{g2}} \\
&= \frac{(1+n_t) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}{(1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}
\end{aligned}
$$

which is higher than the initial level since the baby boomers are now aged $g3$.

The pension surplus in time $t+2$ is given by:[11]

$$
\begin{aligned}
S_{t+2} &= \frac{\tau \cdot w_{t+2} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot (\overline{n} - n_t) \\
&< 0 \qquad (17)
\end{aligned}
$$

Thus, in time $t+2$, the pension system has a deficit since the old-age dependency ratio is higher than the initial level and, thus, population is ageing.

In time $t+3$, the pension surplus is given by:

$$
S_{t+3} = \tau \cdot w_{t+3} \cdot (N_{g1,t+3} + N_{g2,t+3}) - (P_{g3,t+3} \cdot N_{g3,t+3} + P_{g4,t+3} \cdot N_{g4,t+3}) \qquad (18)
$$

[11] For the mathematical details, see Appendix 1d.

where:

$$
\begin{aligned}
N_{g1,t+3} &= N_{g1,t+2} \cdot (1+\overline{n}) = N_{g1,t-1} \cdot (1+\overline{n})^4 \\
N_{g2,t+3} &= N_{g1,t+2} \cdot \overline{\Omega}_{g2} = N_{g1,t-1} \cdot (1+\overline{n})^3 \cdot \overline{\Omega}_{g2} \\
N_{g3,t+3} &= N_{g1,t+1} \cdot \overline{\Omega}_{g3} = N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot \overline{\Omega}_{g3} \\
N_{g4,t+3} &= N_{g1,t} \cdot \overline{\Omega}_{g4} = N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g4}
\end{aligned}
$$

and:

$$
\begin{aligned}
P_{g3,t+3} &= w_{t+2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
P_{g4,t+3} &= P_{g3,t+2} = w_{t+1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}
\end{aligned}
$$

In time $t+3$, the old-age dependency ratio is equal to:

$$
\begin{aligned}
DR_{t+3} &= \frac{N_{g3,t+3} + N_{g4,t+3}}{N_{g1,t+3} + N_{g2,t+3}} = \frac{N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot \overline{\Omega}_{g3} + N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g4}}{N_{g1,t-1} \cdot (1+\overline{n})^4 + N_{g1,t-1} \cdot (1+\overline{n})^3 \cdot \overline{\Omega}_{g2}} \\
&= \frac{(1+\overline{n}) \cdot \overline{\Omega}_{g3} + \frac{1+n_t}{1+\overline{n}} \cdot \overline{\Omega}_{g4}}{(1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}
\end{aligned}
$$

which is higher than the initial level since the baby boomers are now aged $g4$.

The pension surplus in time $t+3$ is given by:[12]

$$
S_{t+3} = \frac{\tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot (\overline{n} - n_t) < 0 \qquad (19)
$$

Thus, in time $t+3$, the pension system has a deficit since the old-age dependency ratio is again higher than the initial level.

Starting from time $t+4$, the population is stable. In fact, the change in the fertility rate in t and $t+1$ does not affect the demographic situation in $t+4$. Consequently, starting from $t+4$, the pension system is balanced according to Proposition 1.

Given that the pension system has a positive surplus in t and $t+1$ (i.e. during the baby boom) and a deficit in $t+2$ and $t+3$ (i.e. when the population is ageing), now we analyze how pension reserves evolve over time.

[12]For the mathematical details, see Appendix 1e.

Pension reserves are equal to zero until time t, since all the previous surpluses are nil. In time $t+1$, pension reserves become positive:

$$Res_{t+1} \quad = \quad S_t \quad = \quad \tau \cdot w_t \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) \quad > \quad 0 \tag{20}$$

In time $t+2$, the value of pension reserves depends on the surplus realized in $t+1$ and on the interest on the reserves accumulated in $t+1$. Here, we assume that the remuneration rate on pension reserves is equal to the average wage bill growth rate, i.e. the notional rate of interest R defined in equation 6. Thus, in time $t+2$, pension reserves are given by:[13]

$$\begin{aligned} Res_{t+2} &= Res_{t+1} \cdot (1+R) \; + \; S_{t+1} \\ &= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1 + \overline{n} + \overline{\Omega}_{g2}) \cdot (n_t - \overline{n}) \quad > \quad 0 \end{aligned} \tag{21}$$

In time $t+3$, pension reserves are given by:[14]

$$\begin{aligned} Res_{t+3} &= Res_{t+2} \cdot (1+R) \; + \; S_{t+2} \\ &= \frac{\tau \cdot w_{t+2} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1 + \overline{n} + \overline{\Omega}_{g2}) \cdot \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot (n_t - \overline{n}) \quad > \quad 0 \end{aligned} \tag{22}$$

In time $t+4$, pension reserves are given by:[15]

$$Res_{t+4} \quad = \quad Res_{t+3} \cdot (1+R) \; + \; S_{t+3} \quad = \quad 0 \tag{23}$$

Thus, the level of pension reserves is nil in the long run and the pension system is sustainable.

Q.E.D.

To resume, if the fertility rate first increases (as during the baby boom) and then decreases (as during the baby boost) and if pensions are computed according to actuarial principles as in equation 7, then the pension system has a positive surplus in the first periods (when the dependency ratio is lower than the initial level) and accumulates pension reserves which will be used to cover the pension deficits when population is ageing. In particular, if pension reserves are remunerated at the notional rate of interest R, then pension reserves become nil starting from time $t+4$, i.e. when the old-age dependency ratio comes back to the initial level. The pension system is sustainable and, thus, is perfectly able to face such a demographic shock.

[13]For the mathematical details, see Appendix 1f.
[14]For the mathematical details, see Appendix 1g.
[15]For the mathematical details, see Appendix 1h.

2.4 Demographic shock on survival probabilities

Here, we analyze the effects of a permanent shock on survival probabilities. We consider separately an increase in the conditional probability to be alive at age $g3$ and an increase in the conditional probability to be alive at age $g4$. These two cases are treated separately because in the first case there is no problem of anticipating the shock since $g3$ corresponds to the age at which individuals receive their first pension. In contrast, in the case of an increase in the conditional probability to be alive at age $g4$, it is necessary to make an assumption about whether or not this shock is known in advance and taken into account in the computation of pension benefits.

2.4.1 A permanent increase in ω_{g3}

Proposition 3. In the case of a permanent increase in the conditional probability to be alive at age $g3$ (ω_{g3}), and if this shock is taken into account in the computation of pension benefits according to equation 7, then the pension system is always balanced and, thus, is sustainable.

Proof.

Starting from time t, we assume that the percentage increase in the conditional probability to be alive at age $g3$ is $\Delta\omega_{g3} > 0$. This implies that the unconditional probability to be alive at age $g3$ and at age $g4$ become $\Omega_{g3,t} = \overline{\Omega}_{g3} \cdot (1+\Delta\omega_{g3})$ and $\Omega_{g4,t+1} = \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g3})$, respectively. The level of pension benefits earned by individuals aged $g3$ in time t is:

$$P_{g3,t} \quad = \quad w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{1}{1+\Delta\omega_{g3}} \tag{24}$$

Thus, pension benefits are lower than in the case without the shock because the pension rule takes into account that these individuals will live longer. Note that the fall in the replacement ratio does not imply that the pension system becomes less generous. In fact, the implicit rate of return on contributions remains unchanged and is equal to the notional rate of interest R.

In time t, the pension surplus is given by:

$$S_t \quad = \quad \tau \cdot w_t \cdot (N_{g1,t} + N_{g2,t}) \; - \; (P_{g3,t} \cdot N_{g3,t} + P_{g4,t} \cdot N_{g4,t}) \tag{25}$$

where:

$$
\begin{aligned}
N_{g1,t} &= N_{g1,t-1} \cdot (1+\overline{n}) \\
N_{g2,t} &= N_{g1,t-1} \cdot \overline{\Omega}_{g2} \\
N_{g3,t} &= \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g3} \cdot (1+\Delta\omega_{g3}) \\
N_{g4,t} &= \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4}
\end{aligned}
$$

and:

$$
\begin{aligned}
P_{g3,t} &= w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{1}{1+\Delta\omega_{g3}} \\
P_{g4,t} &= P_{g3,t-1} = w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}
\end{aligned}
$$

The pension surplus in time t is given by:[16]

$$
S_t = 0 \tag{26}
$$

Thus, even though the shock induces an increase in the old-age dependency ratio, the negative demographic effect is perfectly compensated by the reduction in pension benefits perceived by the generation that lives longer.

In time $t+1$, the pension surplus is given by:

$$
S_{t+1} = \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) - (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \tag{27}
$$

where:

$$
\begin{aligned}
N_{g1,t+1} &= N_{g1,t} \cdot (1+\overline{n}) \\
N_{g2,t+1} &= N_{g1,t} \cdot \overline{\Omega}_{g2} \\
N_{g3,t+1} &= \frac{N_{g1,t}}{(1+\overline{n})} \cdot \overline{\Omega}_{g3} \cdot (1+\Delta\omega_{g3}) \\
N_{g4,t+1} &= \frac{N_{g1,t}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g3})
\end{aligned}
$$

[16] For the mathematical details, see Appendix 2a.

and:

$$P_{g3,t+1} = w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{1}{1+\Delta\omega_{g3}}$$

$$P_{g4,t+1} = P_{g3,t} = w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{1}{1+\Delta\omega_{g3}}$$

The pension surplus in time $t+1$ is given by[17]:

$$S_{t+1} = 0$$

Thus, even though the old-age dependency ratio is higher than the level before the shock and also than the level in t, the pension surplus remains nil since the negative demographic effect is perfectly compensated by the reduction in pension benefits perceived by the two generations that live longer.

Starting from $t+1$, the old-age dependency ratio remains constant and, consequently, the pension surplus is nil for all the future periods. Given that all pension surpluses are equal to zero, the level of pension reserves is always nil and the pension system is sustainable.

Q.E.D.

2.4.2 A permanent and perfectly anticipated increase in ω_{g4}

Proposition 4. In the case of a permanent increase in the conditional probability to be alive at age $g4$ (ω_{g4}), if this shock is *perfectly anticipated* and taken into account in the computation of pension benefits according to equation 7, then pension reserves are strictly positive in the long run and the pension system is sustainable.

Proof.

Starting from time $t+1$, we assume that the percentage increase in the conditional survival probability to be alive at age $g4$ is $\Delta\omega_{g4} > 0$. This implies that the unconditional probability to be alive at age $g4$ becomes $\Omega_{g4,t+1} = \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})$. The increase in $\Omega_{g4,t+1}$ is assumed to be perfectly anticipated in time t which implies a reduction, with respect to the situation without the shock, in the level of pension benefits earned by individuals aged $g3$ in time t who will live longer in the future period. In fact:

[17] For the mathematical details, see Appendix 2b.

$$P_{g3,t} = w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \tag{28}$$

As in the case of a shock on ω_{g3}, the fall in the replacement ratio does not imply that the pension system becomes less generous since the implicit rate of return on contributions is still equal to the notional rate of interest R.

In time t, the pension surplus is given by:

$$S_t = \tau \cdot w_t \cdot (N_{g1,t} + N_{g2,t}) - (P_{g3,t} \cdot N_{g3,t} + P_{g4,t} \cdot N_{g4,t}) \tag{29}$$

where:

$$\begin{aligned}
N_{g1,t} &= N_{g1,t-1} \cdot (1+\overline{n}) \\
N_{g2,t} &= N_{g1,t-1} \cdot \overline{\Omega}_{g2} \\
N_{g3,t} &= \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g3} \\
N_{g4,t} &= \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4}
\end{aligned}$$

and:

$$\begin{aligned}
P_{g3,t} &= w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
P_{g4,t} &= P_{g3,t-1} = w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}
\end{aligned}$$

The pension surplus in time t is:[18]

$$\begin{aligned}
S_t &= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \qquad (30)\\
&\cdot \left[\frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}\right] > 0
\end{aligned}$$

Thus, the pension surplus is positive in t. This is explained by the fact that in time t the old-age dependency ratio is the same as before the shock (it will increase starting from $t+1$) while pension benefits earned by people aged $g3$ are reduced according to actuarial principles.

Starting from time $t+1$, given that the increase in the conditional probability to be alive at age $g4$ is assumed to be permanent, the old-age dependency ratio remains constant at a higher level with respect to the situation without the shock.

[18] For the mathematical details, see Appendix 3a.

In time $t+1$, i.e. when the shock is observed, the pension surplus is given by:

$$S_{t+1} \quad = \quad \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) \; - \; (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \tag{31}$$

where:

$$\begin{aligned} N_{g1,t+1} &= N_{g1,t-1} \cdot (1+\overline{n})^2 \\ N_{g2,t+1} &= N_{g1,t-1} \cdot (1+\overline{n}) \cdot \overline{\Omega}_{g2} \\ N_{g3,t+1} &= N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\ N_{g4,t+1} &= \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) \end{aligned}$$

and:

$$\begin{aligned} P_{g3,t+1} &= w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\ P_{g4,t+1} &= P_{g3,t} \;=\; w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \end{aligned}$$

The pension surplus in time $t+1$ is given by:[19]

$$S_{t+1} \quad = \quad 0 \tag{32}$$

In time $t+1$, the pension surplus is nil since for both generations that are retired and that live longer than in the situation without the demographic shock, the value of pension benefits is computed by taking into account the increase in their life expectancy.

Pension reserves are equal to zero until time t, since all the previous surpluses were nil. In time $t+1$, pension reserves are given by:

$$\begin{aligned} Res_{t+1} &= S_t \\ &= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \qquad (33) \\ &\cdot \left[\frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \right] > 0 \end{aligned}$$

Starting from time $t+2$, considering that pension surpluses are nil starting from $t+1$, pension reserves grow at the constant rate R:

$$Res_{t+k+1} \quad = \quad Res_{t+k} \cdot (1+R) \quad > \quad 0 \quad \text{with } k \geq 1 \tag{34}$$

[19]For the mathematical details, see Appendix 3b.

Thus, the long-run level of pension reserves is positive, implying that the pension system is sustainable.

Q.E.D.

2.4.3 A permanent and unanticipated increase in ω_{g4}

Proposition 5. In the case of a permanent increase in the conditional probability to be alive at age $g4$ (ω_{g4}), if this shock is *unanticipated*, then pension reserves are strictly negative in the long run and the pension system is not sustainable.

Proof.

As in the previous section, we consider a permanent increase in the conditional probability to be alive at age $g4$ starting from time $t+1$. In contrast, here, we assume that the shock is not anticipated in t. However, in $t+1$, the demographic shock is observed. For this reason, we assume that the demographic shock is taken into account in the computation of pension benefits of people aged $g3$ starting from $t+1$.

The fact that the shock is unanticipated implies that the level of pension benefits earned by individuals aged $g3$ in time t is computed without considering the future increase in the probability to be alive at age $g4$ and, thus, remains at the same level as without the shock. Here it is important to note that, even though the replacement ratio remains unchanged, the pension system becomes more generous for these individuals since the implicit rate of return on contributions is higher than the notional rate of interest R.

Thus, in time t, nothing happens with respect to the case where there is no demographic shock: pensions are the same and the number of workers and retirees is the same. The pension surplus is thus equal to zero.

In $t+1$, the increase in the probability to be alive at age $g4$ implies an increase in the number of retirees and, thus, an increase in the old-age dependency ratio. In $t+1$, given that the demographic shock is observed, the pension earned by people aged $g3$ is computed by taking into account the increase in life expectancy. In contrast, the pension earned by individuals aged $g4$ (who were aged $g3$ in t) does not take into account the demographic shock.

In time $t+1$, the pension surplus is given by:

$$S_{t+1} = \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) - (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \tag{35}$$

where:

$$\begin{aligned} N_{g1,t+1} &= N_{g1,t-1} \cdot (1+\overline{n})^2 \\ N_{g2,t+1} &= N_{g1,t-1} \cdot (1+\overline{n}) \cdot \overline{\Omega}_{g2} \\ N_{g3,t+1} &= N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\ N_{g4,t+1} &= \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) \end{aligned}$$

and:

$$\begin{aligned} P_{g3,t+1} &= w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\ P_{g4,t+1} &= P_{g3,t} = w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \end{aligned}$$

The pension surplus in time $t+1$ is given by:[20]

$$\begin{aligned} S_{t+1} = \; & \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\ & \cdot \left[\frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \right] < 0 \end{aligned} \tag{36}$$

Thus, the pension system has a deficit in $t+1$ which is explained by the increase in the old-age dependency ratio and by the fact that the pension earned by people aged $g4$ in $t+1$ is computed in a wrong way, i.e. without using correct actuarial principles. In fact, for these individuals, the implicit rate of return on contributions is higher than R.

Starting from time $t+2$, the pension surplus is always equal to zero. In fact, even if the old-age dependency ratio is higher than the level before the demographic shock, for both generations that are retired pension benefits are computed by taking into account the increase in their life expectancy.

Pension reserves are equal to zero until time $t+1$ (since all the previous surpluses were nil) and become negative in $t+2$:

$$Res_{t+2} = S_{t+1} < 0 \tag{37}$$

[20] For the mathematical details, see Appendix 4a.

In the next periods, considering that pension surpluses are nil starting from $t+2$, we have:

$$Res_{t+k+1} \quad = \quad Res_{t+k} \cdot (1+R) \quad < \quad 0 \quad \text{with } k \geq 2 \tag{38}$$

Thus, the pension debt increases over time at the rate R which coincides with the wage bill growth rate and the GDP growth rate (given a Cobb-Douglas technology), implying that the ratio between pension reserves and GDP is constant in the long run. However, considering that the interest rate on pension reserves and the growth rate of the pension debt are both equal to the notional interest rate R, the pension system is not sustainable.

Q.E.D.

2.4.4 A permanent and partially anticipated increase in ω_{g4}

In sections 2.4.2 and 2.4.3 we have shown that in the case of a permanent increase in the conditional probability to be alive at age $g4$ (ω_{g4}) pension reserves are positive in the long run if this shock is perfectly anticipated, while they are negative if the shock is unanticipated. Now we consider the case where the shock is partially anticipated.

Proposition 6. In the case of a permanent increase in the conditional probability to be alive at age $g4$ (ω_{g4}), there exists a scalar χ that lies between 0 and 1 such that if a fraction χ of the increase in the conditional probability to be alive at age $g4$ is anticipated, then the value of pension reserves is nil in the long run and the pension system is sustainable.

Proof.

As in the previous sections, we consider a permanent increase in the conditional probability to be alive at age $g4$ starting from period $t+1$. Here, we assume that the shock is partially anticipated in t. Again, in $t+1$ i.e. when the demographic shock is observed, we assume that the shock is taken into account in the computation of pension benefits of people aged $g3$ starting from $t+1$.

Considering that pension surpluses are nil starting from $t+2$, the value of pension reserves is equal to zero in the long run if its value is equal to zero in $t+2$. The value of pension reserves in $t+2$ is:

$$\begin{aligned} Res_{t+2} &= Res_{t+1} \cdot (1+R) + S_{t+1} \\ &= S_t \cdot (1+R) + S_{t+1} \end{aligned}$$

The value of χ such that $Res_{t+2} = 0$ is:[21]

$$\chi = \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4})}{\beta \cdot [\beta + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})] + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4})} \quad (39)$$

with $\beta = (1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}$.

Q.E.D.

In particular, the value of χ is lower than 0.5,[22] implying that it is sufficient to anticipate half of the shock to guarantee the sustainability of the pension system.

3 A simulation analysis

In this section, we present some numerical simulations using a stylized OLG model (Auerbach and Kotlikoff, 1987) calibrated on German demographic data. As most of the European countries, Germany will suffer the problem of population ageing. The simulations presented here can be defined as "stylized", for two reasons. First, all parameters in the model, except those concerning the demographic evolution, are fixed at a constant level. Second, we assume, the existence of an NDC pension system where pension benefits are computed according to actuarial principles as described in the theoretical part of this article.

In what follows, we first present a brief description of the model and the calibration procedure. Finally, we present the simulation results of the demographic shock. The effects are analyzed by considering different scenarios, i.e. different ways of computing pension benefits in response to the demographic shock. The results, even though they are based on a stylized model, allow to give an idea of the evolution of pension reserves and, consequently, on the sustainability of the pension system, given a realistic demographic shock.

[21] For the mathematical details, see Appendix 5a.
[22] For the mathematical details, see Appendix 5b.

3.1 Description of the OLG model

3.1.1 Consumers

We assume that individuals live between $a = 20$ to $a = 95$ years old. The demographic evolution is described by the following equations:

$$N_{20,t+1} = N_{20,t} \cdot (1 + n_{t+1}) \tag{40}$$

$$N_{a+1,t+1} = N_{a,t} \cdot \omega_{a+1,t+1} \quad \text{for } a \geq 20 \tag{41}$$

where $N_{a,t}$ is the number of individuals aged a at time t, n_{t+1} is a measure of the fertility rate and $\omega_{a+1,t+1}$ is the probability for an individual belonging to the class age a at time t to be alive in $t+1$. Fertility rates and conditional survival probabilities can vary over time even if, of course, they are kept constant to calibrate the model in steady state.

All individuals are supposed to work between 20 and 64 years old and are retired starting from age 65. For each generation, we assume the existence of a representative agent who has to choose the intertemporal profile of consumption by maximizing his expected intertemporal utility subject to his intertemporal budget constraint. The expected intertemporal utility function for an individual aged $a = 20$ at time t is:

$$E[U_t] = E_t \left[\sum_{a=20}^{95} \left(\frac{1}{1+\rho} \right)^{a-20} \cdot \ln c_{a,t+a-20} \cdot \Omega_{a,t+a-20} \right] \tag{42}$$

where $c_{a,t}$ is consumption of the individual aged a in t, ρ is the intertemporal preference rate and $\Omega_{a,t}$ is the unconditional probability to be alive at age a at time t.

Individuals accumulate capital over time and, in the last period, consume all their available resources, i.e. their wealth, the interest, and the pension earned. Assuming the existence of a life insurance sector (Yaari, 1965), the evolution of capital is described by the following equation:

$$k_{a,t} = k_{a-1,t-1} \cdot (1 + r_t) \cdot \frac{\Omega_{a-1,t-1}}{\Omega_{a,t}} + (1 - \tau - \tau^L) \cdot \eta_{a,t} \cdot \widehat{w}_t \cdot z + P_{a,t} \cdot z - c_{a,t} \tag{43}$$

where $k_{a,t}$ is the capital that an individual aged a owns at the end of time t, r_t is the real interest rate, $\Omega_{a,t}$ is the unconditional probability to be alive at age a at time t,[23] τ is the

[23]Note that the expression $(1 + r_t) \cdot \frac{\Omega_{a-1,t-1}}{\Omega_{a,t}}$ implies $(1 + r_t) \cdot \frac{1}{\omega_{a,t}}$, where $\omega_{a,t}$ is the conditional probability to be alive in t for an individual aged $a - 1$ in $t - 1$. The previous expression can be written as $(1 + r_t + r^{\omega}_{a,t})$ where $r^{\omega}_{a,t} = (1 - \omega_{a,t}) \cdot \frac{1+r_t}{\omega_{a,t}}$. Thus, the additional rate of interest paid by the insurance sector ($r^{\omega}_{a,t}$) is slightly different with respect to the mortality probability $(1 - \omega_{a,t})$, as in Yaari (1965), because the model is in discrete time.

pension contribution rate, τ^L is the tax rate applied to labor incomes, $\eta_{a,t} \cdot \widehat{w}_t$ is the gross wage earned by an individual aged a at time t (which depends on the exogenous individual productivity differentiated by age $\eta_{a,t}$ and on the wage per unit of effective labor $\widehat{w}_t$), and $P_{a,t}$ represents pension benefits which are computed as indicated in section 3.1.3. Finally z represents both the employment rate (assumed to be exogenous, constant over time and the same for each age group) and the fraction of old people that earn pension benefits.

The Euler equation describing the optimal consumption path is:

$$\frac{1}{c_{a,t}} = \frac{1}{1+\rho} \cdot E_t \left[\frac{1+r_{t+1}}{c_{a+1,t+1}} \right] \tag{44}$$

3.1.2 Firms

We assume that a representative firm produces one good using labor and capital according to a Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} \cdot \Gamma_t^{1-\alpha} \tag{45}$$

where Y_t represents real GDP, K_t the quantity of capital employed at the beginning of time t, and Γ_t the quantity of labor employed expressed in effective labor units. The demand of labor and capital is determined in order to maximize profits. First order conditions for profit maximization are $\widehat{w}_t = (1-\alpha) \cdot (K_t/\Gamma_t)^{\alpha}$ and $r_t + \delta = \alpha \cdot (K_t/\Gamma_t)^{\alpha-1}$, where δ is the depreciation rate.

3.1.3 The pension system and the public sector

Assuming the existence of an NDC system, the first pension earned by an individual aged $a = 65$ in $t+1$ is computed such that the expected present value of pension benefits is equal to the expected capitalized value of contributions and assuming that pensions are indexed on inflation and thus remain constant over time. Noting by R the notional rate of interest, we have:

$$P_{65,t+1} = \frac{\sum_{a=20}^{64} \tau \cdot \eta_{a,t} \cdot \widehat{w}_t \cdot (1+R)^{64-a} \cdot \Omega_{a,t+a-64}}{\sum_{a=65}^{95} (1+R)^{64-a} \cdot \Omega_{a,t+a-64}} \tag{46}$$

and:

$$P_{a+1,t+1} = P_{a,t} \quad \text{for } a \geq 65 \tag{47}$$

The surplus of the pension system is given by the difference between the contributions earned and the pensions paid:

$$S_t = \sum_a \tau \cdot \eta_{a,t} \cdot \widehat{w}_t \cdot N_{a,t} \cdot z - \sum_a P_{a,t} \cdot N_{a,t} \cdot z \tag{48}$$

Pension reserves evolve over time according to the surplus defined in the previous equation and the interest on current reserves computed at a rate equal to the notional rate of interest R as defined in equation 6:

$$Res_{t+1} = Res_t \cdot (1 + R) + S_t \tag{49}$$

Concerning the government, the public surplus is given by the difference between taxes on labor incomes (on the basis of the exogenous tax rate τ^L) and government outlays (public expenditures G_t and the interest on the public debt B_t which is assumed to be computed using the notional rate of interest R defined in equation 6):

$$S_t^G = \sum_a \tau^L \cdot \eta_{a,t} \cdot \widehat{w}_t \cdot N_{a,t} \cdot z \; - \; (G_t + B_t \cdot R) \tag{50}$$

The public debt B_t evolves over time according to:

$$B_{t+1} = B_t - S_t^G \tag{51}$$

Finally, we assume that pension reserves are used to finance the public debt:[24]

$$Res_t = B_t$$

[24]This hypothesis is necessary since the theoretical results presented in the previous sections require that pension reserves are remunerated at a rate equal to the notional rate of interest R. Thus, in the model, it is necessary to consider a financial asset (here, government bonds) that pays interest at the same rate of return.

3.1.4 Equilibrium

The equilibrium conditions in the goods market, in the capital market and in the labor market are respectively:

$$Y_t = \sum_a c_{a,t} \cdot N_{a,t} + I_t + G_t + NX_t \tag{52}$$

$$K_t = \sum_a k_{a,t-1} \cdot N_{a,t-1} + K_t^{RoW} \tag{53}$$

$$\Gamma_t = \sum_{a \leq 64} \eta_{a,t} \cdot N_{a,t} \cdot z \tag{54}$$

where I_t represents investments, NX_t net exports and K_t^{RoW} the stock of capital owned by foreign residents (if positive) at the beginning of time t. Investments are given by the sum of private savings, public savings and savings with respect to the rest of world.

In some simulations we assume that the economy is a small-open economy. Thus, the interest rate is fixed at an exogenous and constant level $r_t = \overline{r}$. This implies that the capital-labor ratio is constant $K_t/\Gamma_t = \overline{k} = \left(\frac{\overline{r}+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$ and the wage per unit of effective labor is constant $\widehat{w}_t = (1-\alpha) \cdot \overline{k}^{\alpha}$. In contrast, in the last simulations, we assume that the economy is closed. As noted in the theoretical part of the article, the open economy assumption is necessary to analytically solve the model since, given this assumption, wages grow at a constant rate. However, it is important to note that in the context of population ageing the open economy assumption is not suitable. In fact, population ageing induces a fall in the marginal productivity of capital and, if the domestic interest rate is assumed to be fixed at an exogenous level, this leads to important capital outflows which negatively affect economic growth. This result is reasonable if the country under analysis is the only one facing the demographic shock, which is clearly not the case. In fact, all developed countries will face the population ageing problem which will reduce their interest rates.

3.2 Model calibration

The model is solved for each year between 1960 and 2400 and is calibrated assuming that the German economy is on its balanced growth path. This assumption requires that all parameters are constant and the population is stable.

In particular, we use the following parameters. The productivity growth rate g is 1.5%; the parameter α in the production function is 1/3; the depreciation rate is 3%; the tax rate is

22.5%. Consumption represents 70% of GDP, investment 15% of GDP and public expenditures 15% of GDP. The employment rate z, which also represents the fraction of old people that earn pension benefits, is fixed at 70% for each age group. The earning profile by age is fixed using the following expression $\eta_{a,t} = (1+g)^{t-1} \cdot e^{0.07\cdot(a-20)-0.001\cdot(a-20)^2}$.[25]

The fertility rate n is fixed at the level computed as the average percentage increase in the working-age population (i.e. population aged between 20 and 64 years old) between 1960 and 2050. Using German data coming from the Eurostat demographic projections, n is equal to 0.01%. The survival probabilities, differentiated by age, are fixed at the level observed in 1960 using the Eurostat demographic data. Given these hypotheses, the population is stable. In particular, the old-age dependency ratio, computed as the ratio between the number of people aged 65 and more and the number of people aged 20-64, remains constant over time and is equal to 0.25.

Concerning the pension system, we assume that the pension contribution rate is equal to 15% and that, at the aggregate level, both pensions and social contributions represent 10% of GDP. Considering that the average fertility rate is equal to 0.01%, the notional rate of interest, $R = (1+n)\cdot(1+g)-1$, is equal to 1.51% while the (implicit) replacement ratio is 55.8%.

3.3 Scenarios

The simulation OLG model presented before is used to evaluate the effects, for Germany, of a realistic demographic shock. In particular, concerning the survival rates, we use those (observed and predicted) presented by Eurostat between 1960 and 2050. After 2050, we assume that they remain constant. Concerning the fertility rates, we determine the values between 1960 and 2050 in order to closely reproduce the old-age dependency ratio (ratio between people aged 65 and more and people aged between 20 and 64) and the total population aged 20 and more. After 2050, we assume that the fertility rate remains constant at the level corresponding to the average fertility rate, which is 0.01%. Figure 1 shows the level of the fertility rates used in the simulations. Figures 2 and 3 show the goodness of the calibration procedure. Note that the old-age dependency ratio and the total population cannot be perfectly reproduced since migration flows are not considered in the model.

[25]This implies that the individual productivity increases over time thanks to the technological progress and to the increase in experience, according to a quadratic form.

We simulate six scenarios. In the first four scenarios, as in the theoretical part of the article, we assume that the economy is open and the notional rate of interest is fixed to 1.51% (which equals the average growth rate of the wage bill). The difference between these scenarios concerns the way in which future shocks are modelled. In scenario S1, we assume that the demographic shock is perfectly anticipated in 1960; thus, the survival rates used in the computation of the first pension are those predicted by the demographic projections, and pensions are computed according to correct actuarial principles. Scenario S2 assumes that the future changes in survival rates are not anticipated in 1960; however they are taken into account when they are observed. Scenario S3 assumes that the shock is partially anticipated 1960 and that is taken into account when it is observed. In particular, the fraction of the shock that is anticipated in 1960 is computed such that the value of pension reserves is equal to zero in the long run. Scenario S4 assumes that the shock is not anticipated 1960 and is not taken into account when observed. This implies that the level of replacement ratio never changes.

Note that only in scenario 1, where the change in future survival rates is perfectly anticipated and taken into account in the computation of pension benefits, the implicit rate of return on contributions always coincides with the notional interest rate R. In contrast, in scenarios 2, 3 and 4, the implicit rate of return on contributions is necessarily higher than the rate notional interest rate R since pensions are not computed according to correct actuarial principles.

Clearly, the hypothesis that the demographic shock was perfectly anticipated or even partially anticipated in 1960 can be considered as unrealistic. However, the existence of a population ageing problem was known at least in 1974, as documented by the World Population Conference organized by the United Nations in August 1974. This is why, in scenario 5, we assume that the demographic shock was unanticipated until 1980 and that, starting from 1980, the shock can only be partially anticipated. In particular, assuming that the evolution of the fertility rate is initially unknown, the notional rate of interest used to compute pension benefits is fixed to 2% until 1979 and then to 1.51%. In addition to the hypothesis made in scenario 5, we assume in scenario 6 (i) that the economy is closed (implying that the interest rate is endogenously determined in order to clear the domestic capital market) and (ii) that the interest rate used to compute the interest on pension reserves is equal to the average between the marginal productivity of capital (net of depreciation) and the notional interest rate.

A brief description of the six scenarios is summarized in table 2.

3.4 Simulation results

In this section, we present the simulation results of the demographic shock by focusing on four measures: the ratio of pension surpluses to GDP, the ratio of pension reserves to GDP, the replacement ratio (i.e. the ratio between the first pension and the last wage) and the implicit rate of return on contributions. First of all, we present the results concerning the first four scenarios in which the economy is assumed to be open.

Table 3 shows the effect on pension surpluses (or deficits). In the first three scenarios, where the the notional rate of interest is equal to 1.51% and the survival probabilities are updated at least when the change in survival probabilities is observed, the pension system has surpluses during the baby boom and deficits between 2030 and 2070 when the population ageing problem is more severe. Then, after a period of low surpluses, the pension system become perfectly balanced starting from 2120. In scenario 4, where the survival probabilities are never updated, the pension system has important deficits in particular starting from 2030.

Table 4 shows the effect on the value of pension reserves. In the first scenario, where the shock on the survival probabilities is perfectly anticipated, the pension system accumulates pension reserves which will be used to cover the pension deficits obtained between 2030 and 2070. In the long run, the value of pension reserves is positive and represents 46% of GDP. Thus, as shown in the theoretical part of the article, the pension system is sustainable and perfectly able to face the population ageing problem. In scenario 2, where the future shock on the survival probabilities is not anticipated in 1960 but the change is taken into account when observed, the pension system accumulates reserves until 2030. However, starting from 2030, pension reserves start to decrease and become negative starting from 2060. In the long run pension reserves represent -5.9% of GDP implying that the pension system is not sustainable. Thus, updating the survival rates when the shock is observed is not sufficient to guarantee the pension sustainability. The pension system is sustainable only if the future shock on survival rates is (at least partially) anticipated in 1960. In scenario 3, the fraction of the future shock that is anticipated is computed such that pension reserves are equal to zero in the long run. We find that this fraction is equal to 47% implying that the pension system is sustainable if half of the future increase in survival probabilities is anticipated in 1960.

Tables 5 and 6 show the effect on the generosity of the pension system, respectively in terms of the replacement ratio and of the implicit rate of return on contributions. In the first scenario, the replacement ratio gradually decreases from 55.8% (the initial steady-state level) to 30.4% (the final stationary level). Even though the replacement ratio significantly decreases over time, this does not mean that the system is less generous for the future generations. In fact, given that the shock on the survival probabilities is perfectly anticipated, the implicit rate of return on contributions coincides with the notional rate of interest. Thus, all generations obtain exactly the same rate of return, which is 1.51%. In scenarios 2 and 3, where the shock on the survival probabilities is unanticipated or partially anticipated, the generations that retire between 1960 and 2030 receive a higher replacement ratio and a higher implicit rate of return on contributions with respect to scenario 1 and with respect to future generations. Concerning scenario 4, the fact that survival probabilities are never updated in the computation of pension benefits implies that the replacement ratio remains constant. In addition, table 6 clearly shows the reason why the pension system is unsustainable: the implicit rate of return on contributions is always higher than the sustainable level.

The previous numerical results confirm the theoretical result presented in this article: if the demographic shock is perfectly anticipated or at least partially anticipated, then the PAYG system is sustainable. In the last two scenarios, we assume that the demographic shock was unanticipated until 1980 and only partially anticipated starting from 1980. In scenario 5 we find that the pension system is sustainable if 93% of the increase in survival probabilities is anticipated starting from 1980. Thus, the pension system is sustainable if almost the entire shock is anticipated starting from 1980, which may be considered as unrealistic. However, in scenario 6, where we consider two more realistic assumptions (i.e. that the economy is closed and that the interest rate used to compute the interest on pension reserves is equal to the average between the marginal productivity of capital (net of depreciation) and the notional interest rate), the pension system is sustainable if 26% of the future shock is anticipated starting from 1980. This last simulation, which can be considered as the most appropriate to analyze the effect of a realistic demographic shock, shows that the PAYG system is sustainable and thus is able to face the population ageing problem if, starting from 1980, one quarter of the future shock is anticipated.

4 Conclusions

This article shows that the non-sustainability of the PAYG systems and the necessity to implement drastic reforms that will considerably reduce the well-being of current and future generations are explained by the fact that pension benefits have not been computed according to actuarial principles and, consequently, these countries did not create sufficient pension reserves during the baby boom. It is important to highlight, first, that the creation of pension reserves does not mean that the pension system is funded. In fact, in this article, current pensions are still paid using current contributions, implying that the pension system is still a PAYG system. Only if current contributions are not sufficient to finance pensions, as during the baby boost, then pension reserves can be used to cover pension deficits. Second, even if in a PAYG system current contributions are used to pay current pensions, this does not mean that the entire amount of current contributions has to be used to pay current pensions. The idea of the article is that a fundamental objective of the pension system, in addition to other objectives like consumption smoothing, income security for the elderly and poverty relief of the elderly, is to be actuarially fair in order to treat each generation in the same way without penalizing future generations. In practice, the implicit rate of return on contributions must be the same for all individuals and determined such that the pension system is sustainable. This implies that during the baby boom PAYG systems should have created pension reserves instead of using the entire amount of contributions to pay pensions.

It is thus possible to conclude that the problem of PAYG systems in most European countries is explained by the fact that, during the previous decades, the pension policies were inappropriate. In particular, the political decision to not create (or to create insufficient) pension reserves and to give excessively high implicit rate of return on contributions to previous generations and to generations that are currently retired are responsible of the non-sustainability of PAYG systems. Thus, it is legitimate to ask whether it is fair that future generations, who are clearly not represented in the political decision-making, will have to pay for this political mistake.

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Table 1: Public pension reserves in 2008

	Level[1,2]	**% of GDP**
Australia	59 624	4.7%
Canada	108 917	6.6%
France	77 131	3.9%
Ireland	16 142	8.6%
Japan	117 628 568	22.6%
Korea	235 424 700	21.3%
New Zealand	14 178	7.5%
Norway	89 765	3.4%
Poland	4 445	0.3%
Portugal	8 339	4.7%
Spain	57 223	5.1%
Sweden	735 204	21.7%
United States	2 418 658	16.4%

Source: OECD (https://stats.oecd.org/Index.aspx?DataSetCode=PPRF#)

[1] Data in millions of national currency

[2] Data include sovereign pension reserve fund and social security reserve fund

Figure 1: Fertility rates used in the model

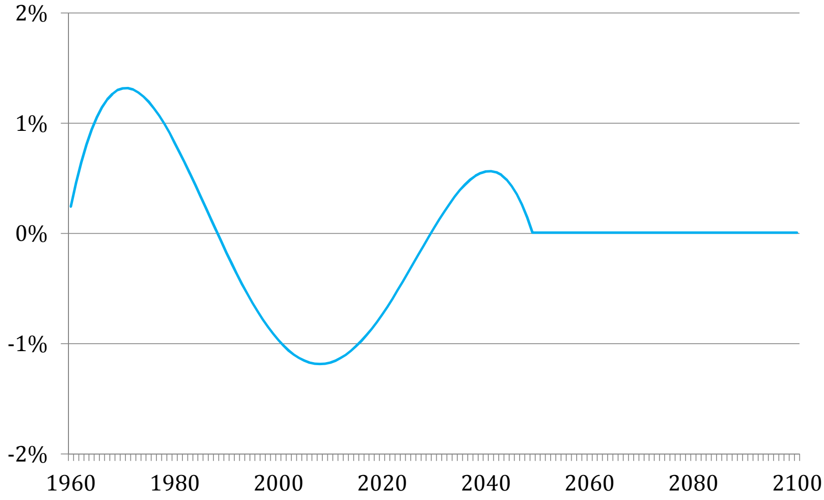

Figure 2: Old age dependency ratio (>65 / 20-64) in Germany

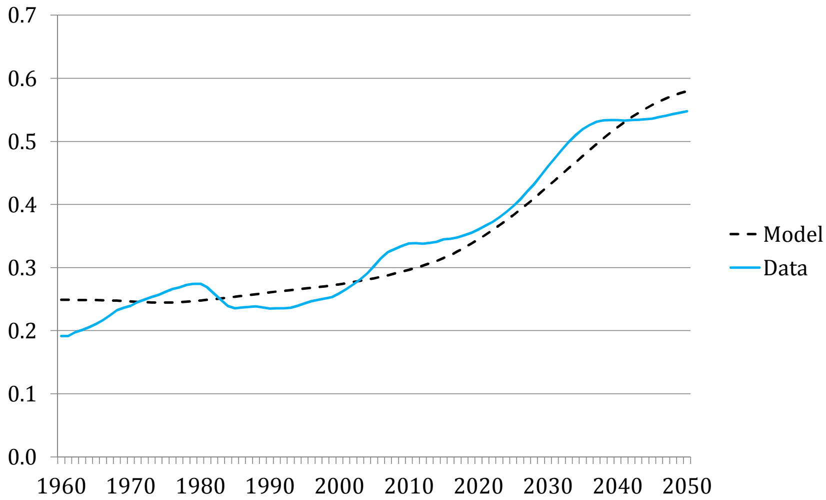

Figure 3: Total popoulation (aged 20 and more) in Germany

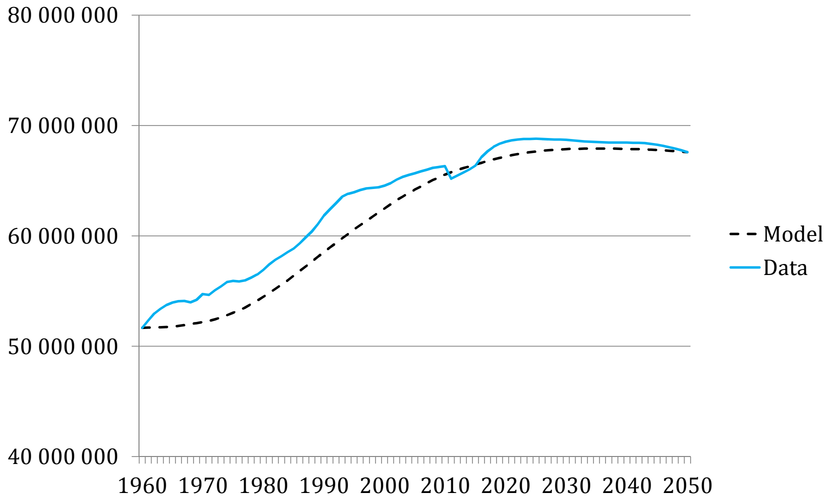

Table 2: Description of the scenarios used in the numerical simulations

	Open economy	Notional rate of interest R	Anticipation of the future increase in survival probabilities	The increase in survival probabilities is taken into account when observed
S1	Yes	1.51%	Perfect	Yes
S2	Yes	1.51%	No	Yes
S3	Yes	1.51%	Partial (47%)	Yes
S4	Yes	1.51%	No	No
S5	Yes	2% until 1979; then 1.51%	No until 1979; then, partial (93%)	Yes
S6	No	2% until 1979; then 1.51%	No until 1979; then, partial (26%)	Yes

Table 3: Pension surplus / GDP

	S1	S2	S3	S4	S5	S6
1960	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
1970	0.4%	0.1%	0.2%	0.1%	-0.6%	-0.7%
1980	1.0%	0.2%	0.4%	-0.1%	-0.4%	-0.8%
1990	1.4%	0.5%	0.6%	-0.7%	0.7%	0.3%
2000	1.8%	1.1%	1.1%	-0.9%	1.5%	1.2%
2010	2.1%	1.5%	1.5%	-1.5%	1.9%	1.7%
2020	1.7%	1.1%	1.1%	-3.0%	1.4%	1.4%
2030	0.4%	-0.1%	-0.1%	-6.0%	0.1%	0.4%
2040	-1.3%	-1.6%	-1.6%	-9.5%	-1.5%	-1.0%
2050	-2.2%	-2.3%	-2.3%	-11.7%	-2.3%	-1.9%
2060	-1.8%	-1.9%	-1.9%	-11.4%	-1.9%	-1.7%
2070	-0.7%	-0.7%	-0.7%	-9.5%	-0.7%	-0.8%
2080	0.2%	0.2%	0.2%	-7.9%	0.2%	0.1%
2090	0.5%	0.5%	0.5%	-7.5%	0.5%	0.4%
2100	0.2%	0.2%	0.2%	-7.9%	0.2%	0.2%
2110	0.1%	0.1%	0.1%	-8.2%	0.1%	0.1%
2120	0.0%	0.0%	0.0%	-8.3%	0.0%	0.0%
2130	0.0%	0.0%	0.0%	-8.3%	0.0%	0.0%
2140	0.0%	0.0%	0.0%	-8.3%	0.0%	0.0%
2150	0.0%	0.0%	0.0%	-8.3%	0.0%	0.0%

Table 4: Pension reserves / GDP

	S1	S2	S3	S4	S5	S6
1960	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
1970	1.5%	0.2%	0.8%	0.1%	-3.7%	-4.4%
1980	8.3%	1.4%	4.2%	0.4%	-9.8%	-14.6%
1990	19.1%	4.1%	8.8%	-3.4%	-7.8%	-20.6%
2000	32.8%	10.9%	16.0%	-10.8%	3.4%	-16.5%
2010	49.9%	22.5%	27.5%	-21.2%	19.9%	-3.7%
2020	68.9%	36.1%	41.1%	-41.0%	37.1%	13.7%
2030	83.4%	43.9%	49.1%	-85.5%	47.4%	28.1%
2040	85.9%	39.2%	44.8%	-169.4%	44.3%	33.0%
2050	73.4%	21.5%	27.4%	-288.0%	27.4%	25.8%
2060	54.0%	0.0%	6.1%	-412.6%	6.1%	11.1%
2070	40.5%	-13.1%	-7.1%	-513.3%	-7.1%	-1.3%
2080	37.5%	-15.1%	-9.3%	-588.6%	-9.2%	-5.8%
2090	40.9%	-11.2%	-5.4%	-657.6%	-5.3%	-4.5%
2100	44.5%	-7.6%	-1.8%	-732.8%	-1.8%	-2.0%
2110	46.0%	-6.1%	-0.3%	-812.1%	-0.3%	-0.7%
2120	46.3%	-5.9%	0.0%	-893.6%	0.0%	-0.2%
2130	46.3%	-5.9%	0.0%	-975.6%	0.0%	-0.1%
2140	46.3%	-5.9%	0.0%	-1057.6%	0.0%	0.0%
2150	46.3%	-5.9%	0.0%	-1139.6%	0.0%	0.0%

Table 5: Replacement ratio

	S1	S2	S3	S4	S5	S6
1960	53.5%	55.8%	54.7%	55.8%	63.1%	63.6%
1970	51.5%	56.2%	53.8%	55.8%	60.1%	62.3%
1980	46.3%	50.8%	50.2%	55.8%	46.9%	49.6%
1990	41.6%	45.5%	45.4%	55.8%	42.5%	44.4%
2000	38.8%	41.9%	41.9%	55.8%	39.9%	41.0%
2010	35.6%	38.0%	38.0%	55.8%	36.6%	37.3%
2020	33.7%	35.4%	35.4%	55.8%	34.8%	34.6%
2030	32.0%	32.8%	32.8%	55.8%	32.7%	31.7%
2040	31.0%	31.2%	31.2%	55.8%	31.2%	30.1%
2050	30.6%	30.6%	30.6%	55.8%	30.6%	30.0%
2060	30.5%	30.5%	30.5%	55.8%	30.5%	30.5%
2070	30.4%	30.4%	30.4%	55.8%	30.4%	30.9%
2080	30.4%	30.4%	30.4%	55.8%	30.4%	30.9%
2090	30.4%	30.4%	30.4%	55.8%	30.4%	30.6%
2100	30.4%	30.4%	30.4%	55.8%	30.4%	30.3%
2110	30.4%	30.4%	30.4%	55.8%	30.4%	30.3%
2120	30.4%	30.4%	30.4%	55.8%	30.4%	30.4%
2130	30.4%	30.4%	30.4%	55.8%	30.4%	30.4%
2140	30.4%	30.4%	30.4%	55.8%	30.4%	30.4%
2150	30.4%	30.4%	30.4%	55.8%	30.4%	30.4%

Table 6: Implicit rate of return on contributions

	S1	S2	S3	S4	S5	S6
1960	1.51%	1.66%	1.59%	1.66%	2.10%	2.13%
1970	1.51%	1.82%	1.67%	1.79%	2.06%	2.21%
1980	1.51%	1.84%	1.79%	2.16%	1.55%	1.81%
1990	1.51%	1.81%	1.81%	2.50%	1.58%	1.81%
2000	1.51%	1.76%	1.76%	2.72%	1.60%	1.76%
2010	1.51%	1.72%	1.72%	2.98%	1.60%	1.72%
2020	1.51%	1.67%	1.67%	3.15%	1.62%	1.67%
2030	1.51%	1.59%	1.59%	3.31%	1.58%	1.59%
2040	1.51%	1.52%	1.52%	3.41%	1.52%	1.52%
2050	1.51%	1.51%	1.51%	3.46%	1.51%	1.51%
2060	1.51%	1.51%	1.51%	3.47%	1.51%	1.51%
2070	1.51%	1.51%	1.51%	3.47%	1.51%	1.51%
2080	1.51%	1.51%	1.51%	3.48%	1.51%	1.51%
2090	1.51%	1.51%	1.51%	3.48%	1.51%	1.51%
2100	1.51%	1.51%	1.51%	3.48%	1.51%	1.51%
2110	1.51%	1.51%	1.51%	3.48%	1.51%	1.51%
2120	1.51%	1.51%	1.51%	3.48%	1.51%	1.51%
2130	1.51%	1.51%	1.51%	3.48%	1.51%	1.51%
2140	1.51%	1.51%	1.51%	3.48%	1.51%	1.51%
2150	1.51%	1.51%	1.51%	3.48%	1.51%	1.51%

TECHNICAL APPENDIX

Appendix 1a.

In absence of shocks the pension surplus in time $t-1$ is given by:

$$
\begin{aligned}
S_{t-1} &= \tau \cdot w_{t-1} \cdot \left(N_{g1,t-1} + \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g2} \right) \\
&- w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g3} \\
&- w_{t-3} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^3} \cdot \overline{\Omega}_{g4} \\
&= \tau \cdot w_{t-1} \cdot \left(N_{g1,t-1} + \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g2} \right) \\
&- \frac{w_{t-1}}{1+\overline{g}} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g3} \\
&- \frac{w_{t-1}}{(1+\overline{g})^2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^3} \cdot \overline{\Omega}_{g4} \\
&= \tau \cdot w_{t-1} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \left(1+\overline{n}+\overline{\Omega}_{g2}\right) \\
&- \tau \cdot w_{t-1} \cdot \frac{(1+\overline{g}) \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\
&- \tau \cdot w_{t-1} \cdot \frac{(1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4} \\
&= \tau \cdot w_{t-1} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \left(1+\overline{n}+\overline{\Omega}_{g2}\right) \cdot \left[1 - \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}\right] \\
&= 0
\end{aligned}
$$

Appendix 1b.

In the case of an increase in the fertility rate in t and a decrease in the fertility rate in $t+1$, the pension surplus in time t is given by:

$$
\begin{aligned}
S_t &= \tau \cdot w_t \cdot (N_{g1,t} + N_{g2,t}) \; - \; (P_{g3,t} \cdot N_{g3,t} + P_{g4,t} \cdot N_{g4,t}) \\[2em]
&= \tau \cdot w_t \cdot (N_{g1,t-1} \cdot (1+n_t) + N_{g1,t-1} \cdot \overline{\Omega}_{g2}) \\
&- \; w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g3} \\
&- \; w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4} \\[2em]
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+n_t+\overline{\Omega}_{g2}) \\
&- \; \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{\overline{\Omega}_{g4} + \overline{\Omega}_{g3} \cdot (1+\overline{n}) \cdot (1+\overline{g}) + \overline{\Omega}_{g4}} \\[2em]
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+n_t+\overline{\Omega}_{g2}) - \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\[2em]
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) \;\; > \;\; 0
\end{aligned}
$$

Appendix 1c.

In the case of an increase in the fertility rate in t and a decrease in the fertility rate in $t+1$, the pension surplus in time $t+1$ is given by:

$$
\begin{aligned}
S_{t+1} &= \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) \;-\; (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \\
\\
&= \tau \cdot w_{t+1} \cdot [N_{g1,t-1} \cdot (1+\overline{n})^2 + N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g2}] \\
&- w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\
&- w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4} \\
\\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot [(1+\overline{n})^2 + (1+n_t) \cdot \overline{\Omega}_{g2}] \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n})^2 \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n}) \cdot \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
\\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot [(1+\overline{n})^2 + (1+n_t) \cdot \overline{\Omega}_{g2}] \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot (1+\overline{n}) \cdot \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
\\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot [(1+\overline{n})^2 + (1+n_t) \cdot \overline{\Omega}_{g2} - (1+\overline{n})^2 - (1+\overline{n}) \cdot \overline{\Omega}_{g2}] \\
\\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) \cdot \overline{\Omega}_{g2} \;>\; 0
\end{aligned}
$$

Appendix 1d.

In the case of an increase in the fertility rate in t and a decrease in the fertility rate in $t+1$, the pension surplus in time $t+2$ is given by:

$$
\begin{aligned}
S_{t+2} &= \tau \cdot w_{t+2} \cdot (N_{g1,t+2} + N_{g2,t+2}) \; - \; (P_{g3,t+2} \cdot N_{g3,t+2} + P_{g4,t+2} \cdot N_{g4,t+2}) \\
&= \tau \cdot w_{t+2} \cdot [N_{g1,t-1} \cdot (1+\overline{n})^3 + N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot \overline{\Omega}_{g2}] \\
&- w_{t+1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g3} \\
&- w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot N_{g1,t-1} \cdot \overline{\Omega}_{g4} \\
&= \tau \cdot w_{t+2} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&- \tau \cdot w_{t+2} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+n_t) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&= \tau \cdot w_{t+2} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \left[1 - \frac{(1+n_t) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}\right] \\
&= \tau \cdot w_{t+2} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} \cdot \frac{\overline{n} - n_t}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}
\end{aligned}
$$

Appendix 1e.

In the case of an increase in the fertility rate in t and a decrease in the fertility rate in $t+1$, the pension surplus in time $t+3$ is given by:

$$
\begin{aligned}
S_{t+3} \;&=\; \tau \cdot w_{t+3} \cdot (N_{g1,t+3} + N_{g2,t+3}) \;-\; (P_{g3,t+3} \cdot N_{g3,t+3} + P_{g4,t+3} \cdot N_{g4,t+3}) \\
&=\; \tau \cdot w_{t+3} \cdot \left[N_{g1,t-1} \cdot (1+\overline{n})^4 + N_{g1,t-1} \cdot (1+\overline{n})^3 \cdot \overline{\Omega}_{g2}\right] \\
&-\; w_{t+2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot \overline{\Omega}_{g3} \\
&-\; w_{t+1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot N_{g1,t-1} \cdot (1+n_t) \cdot \overline{\Omega}_{g4} \\
&=\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^3 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&-\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot \frac{(1+\overline{n})^4 \cdot (1+\overline{g}) \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \overline{\Omega}_{g3} \\
&-\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot \frac{(1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot (1+n_t) \cdot \overline{\Omega}_{g4} \\
&=\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^3 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&-\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n})^2 \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + (1+n_t) \cdot \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&=\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \left[(1+\overline{n}) - \frac{(1+\overline{n})^2 \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + (1+n_t) \cdot \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}\right] \\
&=\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&\cdot\; \frac{(1+\overline{n})^2 \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + (1+\overline{n}) \cdot \overline{\Omega}_{g4} - (1+\overline{n})^2 \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} - (1+n_t) \cdot \overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&=\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \overline{\Omega}_{g4} \cdot \frac{\overline{n} - n_t}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \;<\; 0
\end{aligned}
$$

Appendix 1f.

In the case of an increase in the fertility rate in t and a decrease in the fertility rate in $t+1$, pension reserves in time $t+2$ are given by:

$$
\begin{aligned}
Res_{t+2} &= Res_{t+1} \cdot (1+R) \; + \; S_{t+1} \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) \cdot (1+R) + \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) \cdot \overline{\Omega}_{g2} \\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) \cdot (1+\overline{n}) + \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (n_t - \overline{n}) \cdot \overline{\Omega}_{g2} \\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot (n_t - \overline{n}) \;\; > \;\; 0
\end{aligned}
$$

Appendix 1g.

In the case of an increase in the fertility rate in t and a decrease in the fertility rate in $t+1$, pension reserves in time $t+3$ are given by:

$$
\begin{aligned}
Res_{t+3} &= Res_{t+2}\cdot(1+R)\ +\ S_{t+2}\\[2em]
&= \tau\cdot w_{t+1}\cdot N_{g1,t-1}\cdot(1+\overline{n}+\overline{\Omega}_{g2})\cdot(n_t-\overline{n})\cdot(1+R)\\
&+\ \tau\cdot w_{t+2}\cdot N_{g1,t-1}\cdot(1+\overline{n})^2\cdot(1+\overline{n}+\overline{\Omega}_{g2})\cdot(1+\overline{g})\cdot\overline{\Omega}_{g3}\cdot\frac{\overline{n}-n_t}{(1+\overline{n})\cdot(1+\overline{g})\cdot\overline{\Omega}_{g3}+\overline{\Omega}_{g4}}\\[2em]
&= \tau\cdot w_{t+2}\cdot N_{g1,t-1}\cdot(1+\overline{n}+\overline{\Omega}_{g2})\cdot(n_t-\overline{n})\cdot(1+\overline{n})\\
&-\ \tau\cdot w_{t+2}\cdot N_{g1,t-1}\cdot(1+\overline{n})^2\cdot(1+\overline{n}+\overline{\Omega}_{g2})\cdot(1+\overline{g})\cdot\overline{\Omega}_{g3}\cdot\frac{n_t-\overline{n}}{(1+\overline{n})\cdot(1+\overline{g})\cdot\overline{\Omega}_{g3}+\overline{\Omega}_{g4}}\\[2em]
&= \tau\cdot w_{t+2}\cdot N_{g1,t-1}\cdot(1+\overline{n}+\overline{\Omega}_{g2})\cdot(n_t-\overline{n})\cdot(1+\overline{n})\cdot\left[1-\frac{(1+\overline{n})\cdot(1+\overline{g})\cdot\overline{\Omega}_{g3}}{(1+\overline{n})\cdot(1+\overline{g})\cdot\overline{\Omega}_{g3}+\overline{\Omega}_{g4}}\right]\\[2em]
&= \tau\cdot w_{t+2}\cdot N_{g1,t-1}\cdot(1+\overline{n})\cdot(1+\overline{n}+\overline{\Omega}_{g2})\cdot\overline{\Omega}_{g4}\cdot\frac{n_t-\overline{n}}{(1+\overline{n})\cdot(1+\overline{g})\cdot\overline{\Omega}_{g3}+\overline{\Omega}_{g4}}\quad>\quad 0
\end{aligned}
$$

Appendix 1h.

In the case of an increase in the fertility rate in t and a decrease in the fertility rate in $t+1$, pension reserves in time $t+4$ are given by:

$$
\begin{aligned}
Res_{t+4} \;&=\; Res_{t+3} \cdot (1+R) \;+\; S_{t+3} \\[2ex]
&=\; \tau \cdot w_{t+2} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \overline{\Omega}_{g4} \cdot \frac{n_t - \overline{n}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot (1+R) \\
&+\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \overline{\Omega}_{g4} \cdot \frac{\overline{n} - n_t}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\[2ex]
&=\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \overline{\Omega}_{g4} \cdot \frac{n_t - \overline{n}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&+\; \tau \cdot w_{t+3} \cdot N_{g1,t-1} \cdot (1+\overline{n})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \overline{\Omega}_{g4} \cdot \frac{\overline{n} - n_t}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\[2ex]
&=\; 0
\end{aligned}
$$

Appendix 2a.

In the case of a permanent increase, starting from time t, in the conditional probability to be alive at age $g3$ ($\omega_{g3,t}$) and if this shock is taken into account in the computation of pension benefits according to equation 7, then the pension surplus in time t is given by:

$$
\begin{aligned}
S_t &= \tau \cdot w_t \cdot (N_{g1,t} + N_{g2,t}) \; - \; (P_{g3,t} \cdot N_{g3,t} + P_{g4,t} \cdot N_{g4,t}) \\
\\
&= \tau \cdot w_t \cdot [N_{g1,t-1} \cdot (1+\overline{n}) + N_{g1,t-1} \cdot \overline{\Omega}_{g2}] \\
&- w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{1}{1+\Delta\omega_{g3}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})} \cdot \overline{\Omega}_{g3} \cdot (1+\Delta\omega_{g3}) \\
&- w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4} \\
\\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&- \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&- \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
\\
&= 0
\end{aligned}
$$

Appendix 2b.

In the case of a permanent increase, starting from time t, in the conditional probability to be alive at age $g3$ ($\omega_{g3,t}$) and if this shock is taken into account in the computation of pension benefits according to equation 7, then the pension surplus in time $t+1$ is given by:

$$
\begin{aligned}
S_{t+1} &= \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) - (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \\
&= \tau \cdot w_{t+1} \cdot [N_{g1,t} \cdot (1+\overline{n}) + N_{g1,t} \cdot \overline{\Omega}_{g2}] \\
&- w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{1}{1+\Delta\omega_{g3}} \cdot \frac{N_{g1,t}}{(1+\overline{n})} \cdot \overline{\Omega}_{g3} \cdot (1+\Delta\omega_{g3}) \\
&- w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{1}{1+\Delta\omega_{g3}} \cdot \frac{N_{g1,t}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g3}) \\
&= \tau \cdot w_{t+1} \cdot N_{g1,t} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&= 0
\end{aligned}
$$

Appendix 3a.

In the case of a permanent increase, starting from time $t+1$, in the conditional probability to be alive at age $g4$ (ω_{g4}) and if this shock is anticipated in t and taken into account in the computation of pension benefits according to equation 7, then the pension surplus in time t is given by:

$$
\begin{aligned}
S_t &= \tau \cdot w_t \cdot [N_{g1,t-1} \cdot (1+\overline{n}) + N_{g1,t-1} \cdot \overline{\Omega}_{g2}] \\
&- w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g3} \\
&- w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4} \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&- \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
&- \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
&- \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&\cdot \left[\frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}\right] > 0
\end{aligned}
$$

Appendix 3b.

In the case of a permanent increase, starting from time $t+1$, in the conditional probability to be alive at age $g4$ (ω_{g4}) and if this shock is anticipated in t and taken into account in the computation of pension benefits according to equation 7, then the pension surplus in time $t+1$ is given by:

$$
\begin{aligned}
S_{t+1} &= \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) \; - \; (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \\
&= \tau \cdot w_{t+1} \cdot \left[N_{g1,t-1} \cdot (1+\overline{n})^2 + N_{g1,t-1} \cdot (1+\overline{n}) \cdot \overline{\Omega}_{g2}\right] \\
&- w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \cdot N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\
&- w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) \\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \cdot \\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&= 0
\end{aligned}
$$

Appendix 4a.

In the case of a permanent increase, starting from time $t+1$, in the conditional probability to be alive at age $g4$ (ω_{g4}) and if this shock is *not* anticipated in t, then the pension surplus in time $t+1$ is given by:

$$
\begin{aligned}
S_{t+1} &= \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) - (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \\
&= \tau \cdot w_{t+1} \cdot [N_{g1,t-1} \cdot (1+\overline{n})^2 + N_{g1,t-1} \cdot (1+\overline{n}) \cdot \overline{\Omega}_{g2}] \\
&- w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \cdot N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\
&- w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) \\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}] \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}] \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
&- \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \\
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&\cdot \left[\frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}\right] < 0
\end{aligned}
$$

Appendix 5a.

Consider a permanent increase, starting from time $t+1$, in the conditional probability to be alive at age $g4$ (ω_{g4}) and assume that a fraction χ of the shock is anticipated in t, where χ is computed such that pension reserves are equal to zero in the long run.

The pension surplus in time t is given by:

$$S_t = \tau \cdot w_t \cdot (N_{g1,t} + N_{g2,t}) - (P_{g3,t} \cdot N_{g3,t} + P_{g4,t} \cdot N_{g4,t})$$

where:

$$\begin{aligned} N_{g1,t} &= N_{g1,t-1} \cdot (1+\overline{n}) \\ N_{g2,t} &= N_{g1,t-1} \cdot \overline{\Omega}_{g2} \\ N_{g3,t} &= \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g3} \\ N_{g4,t} &= \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4} \end{aligned}$$

and:

$$\begin{aligned} P_{g3,t} &= w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} \\ P_{g4,t} &= P_{g3,t-1} = w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}} \end{aligned}$$

In what follows, we define $\beta = (1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}$.

Thus, the pension surplus in time t is given by:

$$\begin{aligned}
S_t &= \tau \cdot w_t \cdot [N_{g1,t-1} \cdot (1+\overline{n}) + N_{g1,t-1} \cdot \overline{\Omega}_{g2}] \\
&- w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g3} \\
&- w_{t-2} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{\beta + \overline{\Omega}_{g4}} \cdot \frac{N_{g1,t-1}}{(1+\overline{n})^2} \cdot \overline{\Omega}_{g4} \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&- \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\beta}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} \\
&- \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4}}{\beta + \overline{\Omega}_{g4}} \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} \\
&- \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4}}{\beta + \overline{\Omega}_{g4}} \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&\cdot \left[\frac{\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4}}{\beta + \overline{\Omega}_{g4}}\right] > 0
\end{aligned}$$

The pension surplus in time $t+1$ is given by:

$$S_{t+1} = \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) - (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1})$$

where:

$$\begin{aligned}
N_{g1,t+1} &= N_{g1,t-1} \cdot (1+\overline{n})^2 \\
N_{g2,t+1} &= N_{g1,t-1} \cdot (1+\overline{n}) \cdot \overline{\Omega}_{g2} \\
N_{g3,t+1} &= N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\
N_{g4,t+1} &= \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})
\end{aligned}$$

and:

$$\begin{aligned}
P_{g3,t+1} &= w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
P_{g4,t+1} &= P_{g3,t} = w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})}
\end{aligned}$$

Starting from time $t+1$, given that the increase in the conditional probability to be alive at age $g4$ is assumed to be permanent, the old-age dependency ratio remains constant even if at a level that is higher with respect to the situation without the shock.

Thus, the pension surplus in time $t+1$ is given by:

$$
\begin{aligned}
S_{t+1} &= \tau \cdot w_{t+1} \cdot (N_{g1,t+1} + N_{g2,t+1}) \; - \; (P_{g3,t+1} \cdot N_{g3,t+1} + P_{g4,t+1} \cdot N_{g4,t+1}) \\[2em]
&= \tau \cdot w_{t+1} \cdot \left[N_{g1,t-1} \cdot (1+\overline{n})^2 + N_{g1,t-1} \cdot (1+\overline{n}) \cdot \overline{\Omega}_{g2}\right] \\
&- \; w_t \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{\beta + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \cdot N_{g1,t-1} \cdot \overline{\Omega}_{g3} \\
&- \; w_{t-1} \cdot \frac{\tau \cdot (1+\overline{n})^2 \cdot (1+\overline{g})^2 \cdot (1+\overline{n}+\overline{\Omega}_{g2})}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} \cdot \frac{N_{g1,t-1}}{1+\overline{n}} \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) \\[2em]
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&- \; \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3}}{\beta + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
&- \; \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} \\[2em]
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{\beta + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} \\
&- \; \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} \\[2em]
&= \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&\cdot \; \left[\frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{\beta + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})}\right] \\
&< \; 0
\end{aligned}
$$

Pension reserves are equal to zero until time t, since all the previous surpluses were nil. In time $t+1$, pension reserves are given by:

$$
\begin{aligned}
Res_{t+1} &= S_t \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&\cdot \; \left[\frac{\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})}{\beta + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4}}{\beta + \overline{\Omega}_{g4}}\right] \; > \; 0
\end{aligned}
$$

In $t+2$, pension reserves are:

$$
\begin{aligned}
Res_{t+2} &= Res_{t+1} \cdot (1+R) + S_{t+1} \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot (1+\overline{n}) \cdot (1+\overline{g}) \\
&\cdot \left[\frac{\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4}}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4}}\right] \\
&+ \tau \cdot w_{t+1} \cdot N_{g1,t-1} \cdot (1+\overline{n}) \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \\
&\cdot \left[\frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{(1+\overline{n}) \cdot (1+\overline{g}) \cdot \overline{\Omega}_{g3} + \overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})}\right] \\
&= \tau \cdot w_t \cdot N_{g1,t-1} \cdot (1+\overline{n}+\overline{\Omega}_{g2}) \cdot (1+\overline{n}) \cdot (1+\overline{g}) \\
&\cdot \left[\frac{\overline{\Omega}_{g4} \cdot (\chi-1) \cdot \Delta\omega_{g4}}{\beta+\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4}}{\beta+\overline{\Omega}_{g4}} + \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{\beta+\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}\right]
\end{aligned}
$$

$Res_{t+2} = 0$ if:

$$
\begin{aligned}
\frac{\overline{\Omega}_{g4} \cdot (1-\chi) \cdot \Delta\omega_{g4}}{\beta+\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} &= \frac{\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})}{\beta+\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})} - \frac{\overline{\Omega}_{g4}}{\beta+\overline{\Omega}_{g4}} \\
\frac{\overline{\Omega}_{g4} \cdot (1-\chi) \cdot \Delta\omega_{g4}}{\beta+\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} &= \frac{\beta \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) + \overline{\Omega}^2_{g4} \cdot (1+\Delta\omega_{g4}) - \beta \cdot \overline{\Omega}_{g4} - \overline{\Omega}^2_{g4} \cdot (1+\Delta\omega_{g4})}{[\beta+\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})] \cdot (\beta+\overline{\Omega}_{g4})} \\
\frac{\overline{\Omega}_{g4} \cdot (1-\chi) \cdot \Delta\omega_{g4}}{\beta+\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} &= \frac{\beta \cdot \overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4}) - \beta \cdot \overline{\Omega}_{g4}}{[\beta+\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})] \cdot (\beta+\overline{\Omega}_{g4})} \\
\frac{\overline{\Omega}_{g4} \cdot (1-\chi) \cdot \Delta\omega_{g4}}{\beta+\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} &= \frac{\beta \cdot \overline{\Omega}_{g4} \cdot \Delta\omega_{g4}}{[\beta+\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})] \cdot (\beta+\overline{\Omega}_{g4})} \\
\frac{(1-\chi)}{\beta+\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})} &= \frac{\beta}{[\beta+\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})] \cdot (\beta+\overline{\Omega}_{g4})}
\end{aligned}
$$

Thus:

$$
(1-\chi) \cdot [\beta+\overline{\Omega}_{g4} \cdot (1+\Delta\omega_{g4})] \cdot (\beta+\overline{\Omega}_{g4}) = \beta \cdot [\beta+\overline{\Omega}_{g4} \cdot (1+\chi \cdot \Delta\omega_{g4})]
$$

Thus:

$$\begin{aligned}
& [\beta + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4})] \cdot (\beta + \overline{\Omega}_{g4}) - \chi \cdot [\beta + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4})] \cdot (\beta + \overline{\Omega}_{g4}) \\
= \; & \beta \cdot (\beta + \overline{\Omega}_{g4}) + \chi \cdot \beta \cdot \overline{\Omega}_{g4} \cdot \Delta\omega_{g4}
\end{aligned}$$

$$\begin{aligned}
& \chi \cdot \left\{ \beta \cdot \overline{\Omega}_{g4} \cdot \Delta\omega_{g4} + [\beta + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4})] \cdot (\beta + \overline{\Omega}_{g4}) \right\} \\
= \; & [\beta + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4})] \cdot (\beta + \overline{\Omega}_{g4}) - \beta \cdot (\beta + \overline{\Omega}_{g4})
\end{aligned}$$

$$\begin{aligned}
& \chi \cdot \left\{ \beta \cdot \overline{\Omega}_{g4} \cdot \Delta\omega_{g4} + \beta \cdot (\beta + \overline{\Omega}_{g4}) + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4}) \right\} \\
= \; & \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4})
\end{aligned}$$

$$\begin{aligned}
& \chi \cdot \left\{ \beta \cdot [\overline{\Omega}_{g4} \cdot \Delta\omega_{g4} + \beta + \overline{\Omega}_{g4}] + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4}) \right\} \\
= \; & \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4})
\end{aligned}$$

The value of χ such that the pension system is sustainable is:

$$\chi = \frac{\overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4})}{\beta \cdot [\beta + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4})] + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4})}$$

Appendix 5b.

Considering that $\chi = \frac{\overline{\Omega}_{g4}\cdot(1+\Delta\omega_{g4})\cdot(\beta+\overline{\Omega}_{g4})}{\beta\cdot[\beta+\overline{\Omega}_{g4}\cdot(1+\Delta\omega_{g4})]+\overline{\Omega}_{g4}\cdot(1+\Delta\omega_{g4})\cdot(\beta+\overline{\Omega}_{g4})}$, the value of χ is lower than 1/2 if:

$$\beta \cdot [\beta + \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4})] \quad > \quad \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4})$$

This implies:

$$\beta^2 + \beta \cdot \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \quad > \quad \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot (\beta + \overline{\Omega}_{g4})$$

$$\beta^2 \quad > \quad \overline{\Omega}_{g4} \cdot (1 + \Delta\omega_{g4}) \cdot \overline{\Omega}_{g4}$$

$$(1 + \overline{n})^2 \cdot (1 + \overline{g})^2 \cdot \overline{\Omega}^2_{g3} \quad > \quad \overline{\Omega}^2_{g4} \cdot (1 + \Delta\omega_{g4})$$

$$(1 + R)^2 \quad > \quad \overline{\omega}^2_{g4} \cdot (1 + \Delta\omega_{g4})$$

which is true because $\overline{\omega}_{g4} \cdot (1 + \Delta\omega_{g4})$ (that represents the probability to be alive at age $g4$ after the shock) cannot be higher than one.