

# Dynamic Preferences for Popular Investment Strategies in Pension Funds

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## Abstract

In this paper, we infer preferences that are consistent with some given dynamic investment strategies. Two popular dynamic strategies in the pension funds industry are considered: a constant proportion portfolio insurance (CPPI) strategy and a life-cycle strategy. In both cases, we are able to infer preferences of the pension fund's manager from her investment strategy, and to exhibit the specific expected utility maximization that makes this strategy optimal at any given time horizon. For example, we show that, in a Black-Scholes market, a CPPI strategy is optimal for a fund manager with HARA utility function, while the life-cycle strategy is optimal for an investor with a SAHARA utility function. We also suggest how to modify these strategies if the financial market follows a more general diffusion process than in the Black-Scholes market.

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## 1. Introduction

This paper studies dynamic investment strategies, popular in the industry, especially in pension funds management. Our goal is to infer preferences for which a given investment strategy appears optimal. Our approach is in contrast to most of the existing literature on pension funds. Typically, research in this area starts by specifying a utility function or an objective to optimize (i.e. investor’s preferences) and a given investment horizon (e.g., retirement age) and then derives the optimal investment strategy for accumulating money. A lot of previous studies focus on the expected utility theory. For example, optimal dynamic asset allocation are obtained by Cairns et al. (2006) in a defined contribution plan for an expected utility maximizer (with power utility or log utility) or in a mean-variance setting by Viceira (2007). Boyle and Tian (2009) derive “optimal” parameters of an equity indexed annuity (typical retirement product) for a given expected utility maximizer. Black and Perold (1992) show that a constant proportion portfolio insurance (CPPI) strategy can maximize expected utility when a HARA utility function is assumed. See also Basak (2002), Bertrand and Prigent (2005) and Prigent (2007). Benninga and Blume (1985) further show that the optimality of a portfolio insurance strategy heavily depends on the investor’s utility function.

Other studies have been using preference-free approaches to assess optimality and compare pension plans and investment strategies. For example, Annaert et al. (2009) and Zagst and Kraus (2011) use stochastic dominance to compare portfolio insurance strategies. In a framework including a lot of practical features (such as periodic charges and contributions as well as borrowing constraints), Graf et al. (2012) introduce a new methodology based on risk-return profiles, i.e. the (forward-looking) probability distribution of benefits, to assess the risk-return profiles of existing retirement products and help comparing products across the insurance industry. Their study is illustrated with CPPI and static option-based strategies, Graf (2012) focuses more specifically on life-cycle funds.

As Viceira (2007) explained, there are essentially two types of pension funds investment strategies: the balanced funds (also called constant-mix strategies) and the life-cycle funds (also called target-date funds). Balanced funds are similar to mutual funds with no investment horizon and built on the idea that a fixed fraction of the funds based on investors’ risk tolerance (and independent of their investment horizon) is allocated to stocks. This is

referred as “risk-based investing” and on average (in the 15 largest balanced funds on the market at the end of 2006), 66% was invested in the stock market and 34% in fixed income instruments (see Pang and Warshawsky (2008)). On the contrary, life-cycle funds are “age-based” with the idea that investors should allocate a larger share of their long-term savings to stocks when they are young and decrease this allocation over time. The asset allocation is typically adjusted to become more conservative as the retirement age of the investor approaches. Life-cycle funds and constant-mix portfolio strategies (which are a special case of CPPI strategies) are often set as default choice of asset allocation in numerous defined contribution schemes or related old age provision products. Incorporating labor, Gomes et al. (2008) show that life-cycle funds are superior and that equities are the preferred asset for young households, with the optimal share of equities generally declining prior to retirement.

In this paper, we propose to study both types of strategies and the link between preferences and investment strategies. We focus on two popular dynamic strategies in the pension funds industry, a dynamic portfolio insurance strategy (CPPI with the constant-mix or balanced strategy obtained as a special case) and a life-cycle strategy. We discuss how these latter strategies can be optimal for an expected utility investor and what changes in utility throughout the lifetime of the investor are consistent with them.

Precisely, our contributions are as follows. We are able to directly relate the type of strategy chosen by the manager with a class of utility functions for which it is optimal, and the choice of parameter with the manager’s risk aversion and market parameters. Changes in the strategy’s parameters can be explained by changes in the manager’s risk aversion level or by changes in market conditions. Specifically, we show that CPPI strategies are optimal portfolio choice for an investor with a HARA utility function at all horizon whereas life-cycle funds are optimal for investors with the SAHARA utility function introduced by Chen et al. (2011). In the Black-Scholes setting, we show how CPPI, constant-mix strategies and life-cycle funds are optimal for some explicit expected utility maximizers at any given time horizon. When the instantaneous Sharpe ratio of the underlying risky asset is stochastic (the financial market is not the Black-Scholes model), we explain how to modify these strategies to account for changes in instantaneous Sharpe ratio and preserve optimality.

The techniques used throughout this paper is based on the recent work of Musiela and Zariphopoulou (2009, 2010, 2011). In the standard expected

utility setting, an optimal investment problem consists of a given fixed investment horizon and a utility function. The optimization of expected utility of terminal wealth among all admissible strategies is then performed to obtain the optimal portfolio selection. The approach of Musiela and Zariphopoulou (2009, 2010, 2011) is quite different: they introduce the so-called “forward performance criterion” which allows to define dynamic preferences based on a dynamic investment strategy.

The rest of the paper is organized as follows. The problem and notation are given in Section 2. In particular we recall the definition of a “forward utility”. Examples of dynamic investment strategies are then studied in full details: the CPPI and constant-mix strategies in Section 3, the life-cycle funds in Section 4. Section 5 concludes.

## 2. Setting

### 2.1. Financial Market

We consider a one-dimensional market<sup>1</sup> with two assets: a risky asset and a risk-free bond with the following dynamics under the physical probability measure  $P$

$$\begin{aligned} dS_t &= S_t(\mu_t dt + \sigma_t dW_t), \quad S_0 > 0, \\ dB_t &= r_t B_t dt, \quad B_0 = 1, \end{aligned} \tag{1}$$

where we assume that  $r_t$ ,  $\mu_t$  and  $\sigma_t$  may be stochastic but are adapted to the filtration generated by the Brownian motion  $W_t$  that drives the price of the risky asset. Similarly as Musiela and Zariphopoulou (2009, 2010, 2011), we assume that the market price of risk (or instantaneous Sharpe ratio) defined by

$$\lambda_t \triangleq \frac{\mu_t - r_t}{\sigma_t}$$

is bounded by a deterministic constant  $c > 0$  (for all  $t > 0$ ,  $|\lambda_t| \leq c$ ).

Without loss of generality, we can assume that all amounts are expressed in terms of present values, equivalently  $B_t$  is used as numéraire. A dynamic

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<sup>1</sup>The extension to a multidimensional market is straightforward and would only complicate the notation. As our objective is to better understand the concept of forward performance criterion and illustrate it on some examples, we will only consider the case when there is one risk-free asset and one risky asset in the financial market. A multidimensional market is considered for example in Musiela and Zariphopoulou (2009).

investment strategy consists of an amount  $\pi_t^0$  at time  $t$  invested in the risk-free asset and an amount  $\pi_t$  invested in the risky asset. Let  $X_t^\pi$  denote the present value of the strategy at time  $t$  when the strategy  $\pi$  is implemented. In particular,

$$X_t^\pi = \pi_t^0 + \pi_t.$$

Since  $B_t$  is used as numéraire, it is clear that

$$\begin{aligned} d\pi_t^0 &= 0, \\ d\pi_t &= \pi_t[(\mu_t - r_t)dt + \sigma_t dW_t] = \sigma_t \pi_t(\lambda_t dt + dW_t). \end{aligned}$$

It follows that

$$dX_t^\pi = d\pi_t = \sigma_t \pi_t(\lambda_t dt + dW_t). \quad (2)$$

We define the set  $\mathcal{A}$  of admissible strategies as

$$\mathcal{A} \triangleq \left\{ \pi : \begin{array}{l} (\pi_t)_t \text{ is a self-financing and adapted strategy} \\ \forall t > 0, E\left(\int_0^t |\sigma_s \pi_s|^2 ds\right) < \infty \end{array} \right\}.$$

## 2.2. Preferences

Musiela and Zariphopoulou (2009, 2010, 2011) propose to model preferences of an investor by a forward performance criterion, that we will also call “forward utility” similarly as Berrier et al. (2010). Let us recall its definition given for example in Musiela and Zariphopoulou (2010) (Definition 1 page 329)

**Definition 2.1** (Forward investment performance or Forward utility). *An  $\mathcal{F}_t$ -adapted process  $U_t(x)$  is a forward investment performance if for  $t \geq 0$  and  $x \in \mathbb{D}$  where  $\mathbb{D}$  is an interval of  $\mathbb{R}$ :*

1.  $x \rightarrow U_t(x)$  is strictly concave and increasing
2. for each  $\pi \in \mathcal{A}$  (i.e. for each attainable  $X_s^\pi$ ), and  $t \geq s$ ,

$$E(U_t(X_t^\pi) | \mathcal{F}_s) \leq U_s(X_s^\pi),$$

3. there exists  $\pi^* \in \mathcal{A}$ , for which for all  $t \geq s$ ,

$$E(U_t(X_t^{\pi^*}) | \mathcal{F}_s) = U_s(X_s^{\pi^*}).$$

Although this setting is not equivalent to the well-known expected utility maximization (see for example Musiela and Zariphopoulou (2010)), there are some connections and the definition of a forward utility is quite natural given the following observations. Assume that the investor has a finite trading horizon  $T$  and maximizes over all admissible strategies

$$v(x, t) \triangleq \sup_{\pi \in \mathcal{A}} E(V(X_T^\pi) | \mathcal{F}_t, X_t^\pi = x)$$

where  $V$  is a concave increasing utility function. Under some general conditions, the value function  $v$  satisfies the dynamic programming principle, that is for  $0 \leq s \leq t \leq T$ ,

$$v(x, s) = \sup_{\pi \in \mathcal{A}} E(v(X_t^\pi, t) | \mathcal{F}_s, X_s^\pi = x).$$

Let  $\pi \in \mathcal{A}$  and  $\pi^*$  be the supremum. Then,  $(v(X_s^\pi, s))_s$  and  $(v(X_s^{\pi^*}, s))_s$  are respectively a supermartingale and a martingale for the filtration  $\mathcal{F}_t$  generated by the financial market.

Musiela and Zariphopoulou (2009, 2010, 2011) develop several examples of correspondence between a forward utility and a dynamic investment strategy. They find sufficient conditions for a forward utility to exist and explain the optimality of a dynamic strategy. This forward utility is always formulated as

$$U_t(x) = u(x, A_t)$$

where  $A_t \triangleq \int_0^t \lambda_s^2 ds, t \geq 0$  may be stochastic. We show how their work can be applied to understand CPPI strategies and life-cycle funds.

### 2.3. Dynamic Investment Strategies

The next sections present two relevant examples for the industry. Our first example (Section 3) is the CPPI strategy. As a special case we will look at the constant-mix strategy. Typically, these strategies do not have a given horizon. We exhibit the corresponding forward utility, and show that at any time  $t$ , it belongs to the family of HARA utilities. We also prove that this forward utility can be interpreted as the utility function that makes the CPPI strategy optimal at any given horizon. Our second example (Section 4) concerns life-cycle funds and more generally strategies that involve reducing the investment in risky asset over some time period. We will discuss how this strategy can be optimal for an expected utility investor at any horizon and

what changes in utility throughout the lifetime of the investor are consistent with such a strategy. In particular, we will show that the SAHARA utility as introduced in Chen et al. (2011) can justify the optimality of some life-cycle funds.

### 3. CPPI & Constant-Mix Strategies

The constant proportion portfolio insurance (CPPI) introduced by Black and Perold (1992) is popular in the insurance industry to manage pension funds and variable annuities (see for example Bernard and Le Courtois (2012)). CPPI is indeed a good way to hedge long-term guarantees when the maturity date is not known in advance or when regulators require the guarantee to be met at all times. Let us first recall what a CPPI strategy is.

#### 3.1. CPPI strategy

The CPPI strategy consists of rebalancing a portfolio between the risky and risk-free assets in order to maintain the value of this portfolio always superior to a predefined floor level  $G_t > 0$  at time  $t \geq 0$ , whilst keeping the possibility of large returns. We assume that the floor behaves dynamically like a savings account compounded at the risk-free rate  $r_t$ :

$$dG_t = G_t r_t dt, \quad G_0 = G.$$

Then, one defines the cushion to be the difference between the portfolio value at time  $t$ ,  $V_t$ , and the floor level,  $G_t = GB_t$  at time  $t$ :

$$\forall t \in [0, T], \quad C_t = V_t - G_t.$$

As we work with present values, it can be rewritten in terms of discounted value of the portfolio,  $X_t$ , and we get

$$\forall t \in [0, T], \quad \frac{C_t}{B_t} = X_t - G.$$

The investment rule is to maintain an exposure to the risky asset proportional to the cushion. In other words, the fund manager defines and chooses at time 0 a number  $m > 0$  called the multiple, and rebalances dynamically its portfolio such as

$$\forall t \in [0, T], \quad \pi_t = m \frac{C_t}{B_t} = m(X_t - G).$$

The amount of risk-free asset is therefore at all times

$$\forall t \in [0, T], \pi_t^0 = X_t - \pi_t,$$

and one can write

$$X_t = G + \frac{C_t}{B_t} = \pi_t + \pi_t^0.$$

The first equality is a virtual decomposition of the discounted portfolio value into the guarantee  $G$  and the discounted cushion, whilst the second equality gives the actual decomposition into risky and risk-free assets as discussed in Section 2.

### 3.2. Forward utility corresponding to the CPPI strategy

To ensure that the CPPI strategy is optimal for an expected utility maximizer at any time horizon in the general market described in Section 2.1, we consider a slightly generalized CPPI strategy with random multiple  $m_t = \frac{\lambda_t/\lambda_0}{\sigma_t/\sigma_0} m$  adapted at any time  $t$  to the information available. In the case of a Black-Scholes model, it corresponds to a standard CPPI strategy with fixed multiple  $m$  (because both  $\lambda_t$  and  $\sigma_t$  are constant).

**Proposition 3.1** (General Case). *The dynamic CPPI investment strategy consisting of*

$$\pi_t^* = \frac{\lambda_t/\lambda_0}{\sigma_t/\sigma_0} m (X_t^* - G) \quad (3)$$

*invested in the risky asset (i.e. a CPPI strategy with an adapted multiple  $\frac{\lambda_t/\lambda_0}{\sigma_t/\sigma_0} m$ ) gives the following portfolio value at time  $t$*

$$X_t^* \triangleq X_t^{\pi^*} = (x - G) \exp \left( \left( \gamma - \frac{1}{2} \gamma^2 \right) A_t + \gamma M_t \right) + G, \quad X_0^* = x > G \quad (4)$$

*where  $\gamma = \sigma_0 m / \lambda_0$ ,  $A_t \triangleq \int_0^t \lambda_s^2 ds$  and  $M_t \triangleq \int_0^t \lambda_s dW_s$ . This portfolio value process corresponds to the optimum for the forward utility  $U_t(x) = u(x, A_t)$  where  $u(x, s)$  is given for  $x \in (G, \infty)$  and  $s \geq 0$  by*

$$u(x, s) = \begin{cases} \frac{\gamma}{\gamma-1} (x - G)^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}s}, & \gamma \in (0, 1) \cup (1, \infty), \\ \ln(x - G) - \frac{s}{2}, & \gamma = 1. \end{cases} \quad (5)$$



*Proof.* See Appendix A. □

We observe that the forward utility  $u(\cdot, s)$  belongs to the HARA family function of utilities at all time  $s$ .

**Proposition 3.2.** *Reciprocally, given any time  $T$ , consider the following portfolio optimization problem to maximize the utility of wealth at time  $T$*

$$\max_{\pi \in \mathcal{A}} E(u(X_T, A_T)),$$

where  $A_T = \int_0^T \lambda_s^2 ds$  and where  $u(\cdot, \cdot)$  is given by (5) and defined over  $(G, \infty) \times [0, \infty)$ . Then the optimal allocation is a dynamic CPPI strategy  $\pi_t^* = \frac{\lambda_t/\lambda_0}{\sigma_t/\sigma_0} m(X_t^* - G)$ .

*Proof.* See Appendix A. □

Proposition 3.2 shows that a dynamic CPPI is the optimal solution to a utility maximization. The optimal strategy (3), dynamic CPPI, in stochastic environment implies that we have to rebalance the investment strategy depending on  $\lambda_t$  and  $\sigma_t$ . It is obvious that we have to adjust the investment strategy dynamically if the market is stochastic and the investment opportunity changes dynamically. We obtain the optimality of the standard CPPI strategy in the Black-Scholes model as a corollary of Propositions 3.1 and 3.2.

**Corollary 3.1** (Black-Scholes Case). *Assume that the risky asset follows a Black-Scholes model (i.e.  $\mu$ ,  $r$  and  $\sigma$  are constant and  $\lambda \triangleq (\mu - r)/\sigma$ ). Define  $\gamma = \sigma m/\lambda$ . Then, we have the following results.*

- With the CPPI strategy  $\pi_t^* = m(X_t^* - G)$ , the portfolio value is given by

$$X_t^* = (x - G) \exp\left(\left(\sigma \lambda m - \frac{1}{2} \sigma^2 m^2\right) t + \sigma m W_t\right) + G, \quad X_0^* = x > G$$

and the corresponding forward utility is  $U_t(x) = u(x, \lambda^2 t)$  with  $u(\cdot, \cdot)$  is given by (5).

- Given any time  $T$ , consider the following portfolio optimization problem to maximize the utility of wealth at time  $T$

$$\max_{\pi \in \mathcal{A}} E(u(X_T, \lambda^2 T)),$$

with  $u(\cdot, \cdot)$  given by (5). Then the optimal allocation is CPPI strategy  $\pi_t^* = m(X_t^* - G)$  where the multiple is  $m = \frac{\lambda\gamma}{\sigma}$ .

It is known that in the Black-Scholes setting, the portfolio value of a CPPI strategy  $X$  is a shifted lognormal process (see Prigent (2007)). Note that  $m$  is an indicator of the riskiness of the strategy. It is clear that an increased risk aversion decreases  $\gamma$  and thus  $m$ . An increase in the volatility  $\sigma$  decreases  $m$ .

#### 4. Life-Cycle Funds

In recent years, most pension plans have defined contribution (DC) but no defined benefits (DB). An employee with a DC pension plan has to make her own savings and investment decisions to fund her income after retirement. But, as Viceira (2007) pointed out, a large number of DC plan participants usually adopt the investment strategy suggested by the DC plan sponsor as a default option and rarely rebalance their portfolio by themselves. A typical (and popular) asset allocation strategy for managing equity risk during the accumulation phase of a defined contribution (DC) pension plan is the “life-cycle funds”. At the beginning of the plan, the contributions are invested entirely (or almost entirely) in equities. Then on a predetermined date prior to retirement (e.g., ten years), the assets are switched gradually into bonds at a rate equal to the inverse of the length of the switchover period (e.g., 10% per year). By the date of retirement, all (or almost all) the assets are held in bonds, which are then sold to purchase a life annuity to provide the pension payments. Cairns et al. (2006) extend this strategy to “stochastic lifestyling” and compare it against various static and deterministic life-cycle funds.

In this section, we present the Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA) class of utility functions introduced by Chen et al. (2011) and give the corresponding forward utility and optimal strategy. It is shown that this optimal strategy displays the age-based investing feature of life-cycle funds which means that the optimal investment in risky asset is a decreasing function of time.

#### 4.1. SAHARA utility functions and related forward utility

A SAHARA utility function is given by  $U(x)$ ,  $x \in \mathbb{R}$ , whose absolute risk aversion  $\gamma_A(x) = -U_{xx}(x)/U_x(x)$  satisfies

$$\gamma_A(x) = \frac{1}{\sqrt{a^2(x-d)^2 + c^2}},$$

with  $a > 0$ ,  $c > 0$  and  $d \in \mathbb{R}$ . We can assume that  $d = 0$  without loss of generality. Then  $U(x)$  is given as follows (up to a linear transformation)

- If  $a = 1$ ,

$$U(x) = \frac{1}{2} \ln \left( x + \sqrt{x^2 + c^2} \right) + \frac{1}{2c^2} x \left( \sqrt{x^2 + c^2} - x \right). \quad (6)$$

- If  $a \neq 1$ ,

$$U(x) = \frac{a(a+1) \left( ax^2 + x\sqrt{a^2x^2 + c^2} \right) + c^2}{(a^2 - 1) \left( ax + \sqrt{a^2x^2 + c^2} \right)^{1 + \frac{1}{a}}}. \quad (7)$$

The formula for the utility function can be obtained from Proposition 2.2 in Chen et al. (2011). Details are given in Appendix B.

An important feature of SAHARA utility functions is that they give closed form convex dual  $\tilde{U}(y) \triangleq \sup_{x \in \mathbb{R}} (U(x) - xy)$  and the inverse marginal utility  $I(y) \triangleq (U_x)^{-1}(y)$ . The next proposition exhibits a dynamic strategy  $\pi_t^*$  and corresponding portfolio value  $X_t^*$  with decreasing allocation to the risky asset over time and the corresponding forward utility at  $t$  for which it is an optimal strategy.

**Proposition 4.1** (General Case). *Consider an investment strategy*

$$\pi_t^* = \frac{\lambda_t}{\sigma_t} \sqrt{a^2(X_t^*)^2 + b^2 e^{-a^2 A_t}}, \quad (8)$$

then  $\pi_t^*$  gives the following portfolio value at time  $t$

$$X_t^* = \frac{b}{a} e^{-\frac{1}{2} a^2 A_t} \sinh \left( \ln \left( \frac{a}{b} x + \sqrt{\frac{a^2}{b^2} x^2 + 1} \right) + a A_t + a M_t \right), \quad x \in \mathbb{R}, \quad (9)$$

where  $a > 0$ ,  $b > 0$ ,  $A_t = \int_0^t \lambda_s^2 ds$ , and  $M_t = \int_0^t \lambda_s dW_s$ . Then the corresponding forward utility  $U_t(x) = u(x, A_t)$  where  $u(x, s)$  is defined over  $\mathbb{R} \times [0, \infty)$  and given by

- If  $a = 1$ ,

$$u(x, s) = \frac{1}{2} \ln(x + \sqrt{x^2 + b^2 e^{-s}}) + \frac{e^s}{2b^2} x (\sqrt{x^2 + b^2 e^{-s}} - x) - \frac{s}{4}. \quad (10)$$

- If  $a \neq 1$ ,

$$u(x, s) = e^{\frac{1}{2}(1-a)s} \frac{a(a+1)(ax^2 + x\sqrt{a^2x^2 + b^2e^{-a^2s}}) + b^2e^{-a^2s}}{(a^2-1)(ax + \sqrt{a^2x^2 + b^2e^{-a^2s}})^{1+\frac{1}{a}}}. \quad (11)$$

*Proof.* See Appendix A. □

We observe that at any time  $t$ , the forward utilities obtained in (10) and (11) are SAHARA utilities with varying parameters. Proposition 4.2 makes this fact clearer and is needed to discuss the risk aversion of the forward utility naturally.

**Proposition 4.2.** *Given any time  $T$ , consider the following portfolio optimization problem to maximize the utility of wealth at time  $T$*

$$\max_{\pi \in \mathcal{A}} E(u(X_T, A_T)),$$

where  $A_T = \int_0^T \lambda_s^2 ds$  and where  $u(\cdot, \cdot)$  is defined over  $\mathbb{R} \times [0, \infty)$  and given by (10) and (11) for  $a > 0$  and  $b > 0$ . Then the optimal allocation is given by (8).

*Proof.* See Appendix A. □

Let us consider a Black-Scholes market model (with constant  $\mu$ ,  $r$ , and  $\sigma$ ), then we have the following corollary of Propositions 4.1 and 4.2.

**Corollary 4.1** (Black-Scholes Case). *Assume that  $\mu$ ,  $r$ , and  $\sigma$  are constant. Then, we have the following results.*

- The following investment strategy

$$\pi_t^* = \frac{\lambda}{\sigma} \sqrt{a^2(X_t^*)^2 + b^2 e^{-a^2 \lambda^2 t}}, \quad (12)$$

in the risky asset gives the portfolio value at time  $t$  below

$$X_t^* = \frac{b}{a} e^{-\frac{1}{2} a^2 \lambda^2 t} \sinh \left( \ln \left( \frac{a}{b} x + \sqrt{\frac{a^2}{b^2} x^2 + 1} \right) + a \lambda^2 t + a \lambda W_t \right), \quad x \in \mathbb{R},$$

where  $a > 0$  and  $b > 0$ . Moreover, we have a corresponding forward utility  $U_t(x) = u(x, \lambda^2 t)$  where  $u(x, s)$  is given by (10) and (11).

- *Reciprocally, given any time  $T$ , consider the following portfolio optimization problem to maximize the utility of wealth at time  $T$*

$$\max_{\pi \in \mathcal{A}} E(u(X_T, \lambda^2 T)),$$

where  $u(\cdot, \cdot)$  is defined over  $\mathbb{R} \times [0, \infty)$  and given by (10) and (11) for  $a > 0$  and  $b > 0$ . Then the optimal allocation is given by (12).

#### 4.2. Link to life-cycle funds

Let us define the local (absolute) risk aversion function

$$\gamma(x, s) \triangleq -u_{xx}(x, s)/u_x(x, s)$$

from the forward utility  $u(\cdot, \cdot)$ , where  $u(\cdot, \cdot)$  is given by (10) and (11). Then, we have

$$\gamma(x, s) = \frac{1}{\sqrt{a^2 x^2 + b^2 e^{-a^2 s}}}. \quad (13)$$

It is easily seen that the local risk aversion function (13) is an increasing function of  $s$ . If there is an economic agent with a SAHARA utility function, her optimal investment strategy becomes more conservative as time goes because her local risk aversion is an increasing function of time  $s$ . As a consequence, the optimal allocation to the risky asset (12) is a decreasing function of time. So the investment strategy (12) is a life-cycle fund for an agent whose preference is given by SAHARA utility function.

The optimal strategy (8) in the general case shares similar features, but we have to rebalance the investment taking into account  $\lambda_t$  and  $\sigma_t$  because the market is stochastic. This is consistent with Viceira (2007) who suggested that the market conditions should be involved in determining the asset allocation path of life-cycle funds. The standard life-cycle funds, consisting of a linear decrease of the percentage invested in risky asset does not appear optimal. The way to decrease the allocation over time, depends on changes in market conditions and risk aversion.

## 5. Conclusion and Future Research Directions

The study of two popular dynamic investment strategies in the pension funds industry shows that the HARA and SAHARA roles may play a key

role in explaining fund manager's decisions or in modeling optimal decision making. Future research directions include proving the existence and giving an explicit construction of the forward utility for any type of investment strategies. In particular, this would help in explaining how caps and floors (that are often embedded in pension products) may appear optimal.

## Appendix A. Proofs of Propositions

*Proof of Proposition 3.1.* First of all, let us show that the dynamic CPPI investment strategy (3) gives the portfolio value (4). Let  $X_t^* = X_t^{\pi^*}$  be the portfolio value at time  $t$  with dynamic CPPI investment strategy (3) and initial portfolio value  $X_0^* = x > G$ , then  $X_t^*$  satisfies

$$dX_t^* = (X_t^* - G)(\gamma\lambda_t^2 dt + \lambda_t dW_t), \quad X_0^* = x > G,$$

where  $\gamma = \sigma_0 m / \lambda_0$ . Let us define  $\hat{X}_t \triangleq X_t^* - G$ , then  $\hat{X}_t$  is a geometric Brownian motion which satisfies

$$d\hat{X}_t = \hat{X}_t(\gamma\lambda_t^2 dt + \gamma\lambda_t dW_t), \quad \hat{X}_0 = x - G > 0.$$

Hence,  $\hat{X}_t$  is given by

$$\hat{X}_t = \hat{X}_0 \exp\left(\left(\gamma - \frac{1}{2}\gamma^2\right) \int_0^t \lambda_s^2 ds + \gamma \int_0^t \lambda_s dW_s\right), \quad \hat{X}_0 = x - G > 0,$$

and equivalently,  $X_t^*$  is given by (4).

Now we show that  $U_t(x) = u(x, A_t)$ , where  $u(x, s)$  is given by (5) satisfies the conditions to be a forward utility. It is easily seen that  $u(x, s)$  satisfies  $u_x(x, s) > 0$  and  $u_{xx}(x, s) < 0$  for  $x \in (G, \infty)$  and  $s \geq 0$ . Since  $U_t(x)$  is constructed by  $U_t(x) = u(x, A_t)$ , we conclude that  $x \rightarrow U_t(x)$  is strictly concave and increasing. For each strategy  $\pi \in \mathcal{A}$ , by applying Itô's formula to  $U_t(X_t^\pi) = u(X_t^\pi, A_t)$ , we have

$$\begin{aligned} dU_t(X_t^\pi) &= u_x(X_t^\pi, A_t) \sigma_t \pi_t dW_t \\ &\quad + \lambda_t^2 \left[ u_t(X_t^\pi, A_t) + u_x(X_t^\pi, A_t) \alpha_t + \frac{1}{2} u_{xx}(X_t^\pi, A_t) \alpha_t^2 \right] dt, \end{aligned}$$

where  $\alpha_t \triangleq \sigma_t \pi_t / \lambda_t$ . Note that  $u(x, s)$  in (5) satisfies

$$u_s(x, s) = \frac{u_x^2(x, s)}{2u_{xx}(x, s)},$$

for  $x \in (G, \infty)$  and  $s \geq 0$ . Since  $u_{xx}(x, t) < 0$  and  $u(x, t)$  satisfies  $u_t(x, t) =$

$u_x^2(x, t)/2u_{xx}(x, t)$ , we have

$$\begin{aligned}
& u_t(X_t^\pi, A_t) + u_x(X_t^\pi, A_t)\alpha_t + \frac{1}{2}u_{xx}(X_t^\pi, A_t)\alpha_t^2 \\
&= \frac{1}{2}u_{xx}(X_t^\pi, A_t) \left( \alpha_t + \frac{u_x(X_t^\pi, A_t)}{u_{xx}(X_t^\pi, A_t)} \right)^2 + u_t(X_t^\pi, A_t) - \frac{u_x^2(X_t^\pi, A_t)}{2u_{xx}(X_t^\pi, A_t)} \\
&\leq u_t(X_t^\pi, A_t) - \frac{u_x^2(X_t^\pi, A_t)}{2u_{xx}(X_t^\pi, A_t)} = 0,
\end{aligned}$$

and this means that the drift of  $dU_t(X_t^\pi)$  is non-positive. Hence we conclude that

$$E(U_t(X_t^\pi)|\mathcal{F}_s) \leq U_s(X_s^\pi), \quad (\text{A.1})$$

for  $0 \leq s \leq t$ . It remains to show that (A.1) holds as equality with the dynamic CPPI strategy (3). For  $\gamma \in (0, 1) \cup (1, \infty)$ ,  $U_t(X_t^*) = u(X_t^*, A_t)$  can be written as

$$U_t(X_t^*) = \frac{\gamma}{\gamma-1}(x-G)^{\frac{\gamma-1}{\gamma}} \exp\left(-\frac{1}{2}\int_0^t(\gamma-1)^2\lambda_s^2 ds + \int_0^t(\gamma-1)\lambda_s dW_s\right).$$

By applying Itô's formula to  $U_t(X_t^*)$ , we have

$$dU_t(X_t^*) = \gamma(x-G)^{\frac{\gamma-1}{\gamma}} \exp\left(-\frac{1}{2}\int_0^t(\gamma-1)^2\lambda_s^2 ds + \int_0^t(\gamma-1)\lambda_s dW_s\right) \lambda_t dW_t,$$

and this means that the drift of  $dU_t(X_t^*)$  is zero. For  $\gamma = 1$ ,  $U_t(X_t^*)$  can be written as

$$U_t(X_t^*) = \ln(x-G) + \int_0^t \lambda_s dW_s.$$

Thus, for any  $\gamma > 0$ , it is concluded that

$$E(U_t(X_t^*)|\mathcal{F}_s) = U_s(X_s^*),$$

for  $0 \leq s \leq t$  and this completes the proof.  $\square$

*Proof of Proposition 3.2.* For  $x \in (G, \infty)$  and  $0 \leq t \leq T$ , let us consider the following optimization problem

$$V(x, t; T) \triangleq \sup_{\pi \in \mathcal{A}} E(u(X_T, A_T)|\mathcal{F}_t, X_t = x), \quad (\text{A.2})$$



where  $u(x, s)$  is given by (5),  $A_t = \int_0^t \lambda_s^2 ds$ , and  $X_t$  follows (2). Then by the dynamic programming principle,

$$V(x, t; T) = \sup_{\pi \in \mathcal{A}} E(V(X_r^\pi, r; T) | \mathcal{F}_t, X_t^\pi = x),$$

for  $0 \leq t \leq r \leq T$ . Let us assume that there exists an optimal strategy  $\hat{\pi}$  of the optimization problem (A.2), then  $\hat{\pi}$  satisfies

$$V(x, t; T) = E(V(X_r^{\hat{\pi}}, r; T) | \mathcal{F}_t, X_t^{\hat{\pi}} = x),$$

for  $0 \leq t \leq r \leq T$ . Note that  $V(x, T; T) = u(x, A_T)$  by definition of  $V(x, t; T)$ , and that, for  $0 \leq t \leq r$ ,  $E(U_r(X_r^\pi) | \mathcal{F}_t) \leq U_t(X_t^\pi)$  for any  $\pi \in \mathcal{A}$  by definition of forward utility. Therefore, we have

$$\begin{aligned} V(x, t; T) &= E(V(X_T^{\hat{\pi}}, T; T) | \mathcal{F}_t, X_t^{\hat{\pi}} = x) \\ &= E(u(X_T^{\hat{\pi}}, A_T) | \mathcal{F}_t, X_t^{\hat{\pi}} = x) \\ &\leq u(x, A_t). \end{aligned} \tag{A.3}$$

Let  $\pi^*$  be the dynamic CPPI strategy given by (3), then  $\pi^*$  satisfies

$$u(x, A_t) = E(u(X_r^{\pi^*}, A_r) | \mathcal{F}_t, X_t^{\pi^*} = x), \tag{A.4}$$

for  $0 \leq t \leq r$ . By definition of  $V(x, t; T)$ , it is obvious that, for  $0 \leq t \leq r \leq T$ ,  $E(V(X_r^\pi, r; T) | \mathcal{F}_t, X_t^\pi = x) \leq V(x, t; T)$ , for any  $\pi \in \mathcal{A}$ . Therefore, another inequality is derived as

$$\begin{aligned} u(x, A_t) &= E(u(X_T^{\pi^*}, A_T) | \mathcal{F}_t, X_t^{\pi^*} = x) \\ &= E(V(X_T^{\pi^*}, T; T) | \mathcal{F}_t, X_t^{\pi^*} = x) \\ &\leq V(x, t; T). \end{aligned} \tag{A.5}$$

From (A.3) and (A.5), it is concluded that

$$u(x, A_t) = V(x, t; T), \tag{A.6}$$

for  $x \in (G, \infty)$  and  $0 \leq t \leq T$ . Note that we can derive

$$\begin{aligned} V(x, t; T) &= u(x, A_t) \\ &= E(u(X_r^{\pi^*}, A_r) | \mathcal{F}_t, X_t^{\pi^*} = x) \\ &= E(V(X_r^{\pi^*}, r; T) | \mathcal{F}_t, X_t^{\pi^*} = x), \end{aligned}$$

for  $x \in (G, \infty)$  and  $0 \leq t \leq r \leq T$ , by using (A.4) and (A.6). This implies that the dynamic CPPI strategy  $\pi^*$  given by (3) is an optimal strategy of the optimization problem (A.2) and this completes the proof.  $\square$

*Proof of Proposition 4.1.* By applying Itô's formula to  $X_t^*$  given by (9), it can be verified that  $X_t^*$  satisfies

$$dX_t^* = \sigma_t \pi_t^* (\lambda_t dt + dW_t), \quad X_0^* = x,$$

where  $\pi_t^*$  is given by (8). Therefore, we conclude that  $\pi_t^*$  in (8) gives the portfolio value  $X_t^*$  in (9), that is  $X_t^* = X_t^{\pi^*}$ .

It remains to show that  $U_t(x) = u(x, A_t)$ , where  $u(x, s)$  is given by (10) and (11) satisfies the conditions to be a forward utility. For each  $a > 0$ , let us define a function  $y(x, s; a)$  by

$$y(x, s; a) = \frac{a}{b} e^{\frac{1}{2}a^2 s} x + \sqrt{\frac{a^2}{b^2} e^{a^2 s} x^2 + 1},$$

for  $x \in \mathbb{R}$  and  $s \geq 0$ . Then it can be verified that  $u(x, s)$  given by (10), for  $a = 1$ , can be written as

$$u(x, s) = \frac{1}{2} \ln y(x, s; 1) - \frac{1}{4y(x, s; 1)^2} - \frac{1}{2}s + \frac{1}{2} \ln b + \frac{1}{4},$$

and  $u(x, s)$  given by (11), for  $a \neq 1$ , can be written as

$$u(x, s) = \frac{b^{1-\frac{1}{a}}}{a} e^{-\frac{1}{2}(a^2-1)s} \left( \frac{a}{2(a-1)} y(x, s; a)^{\frac{a-1}{a}} - \frac{a}{2(a+1)} y(x, s; a)^{-\frac{a+1}{a}} \right).$$

We observe that, for all  $a > 0$ ,  $u_x(x, s)$ ,  $u_{xx}(x, s)$ , and  $u_s(x, s)$  can be written as

$$u_x(x, s) = b^{-\frac{1}{a}} e^{\frac{1}{2}t} y(x, s; a)^{-\frac{1}{a}}, \quad (\text{A.7})$$

$$u_{xx}(x, s) = -2b^{-1-\frac{1}{a}} e^{(\frac{1}{2}a^2+\frac{1}{2})t} \frac{y(x, s; a)^{1-\frac{1}{a}}}{y(x, s; a)^2 + 1}, \quad (\text{A.8})$$

$$u_t(x, s) = -\frac{1}{4} b^{1-\frac{1}{a}} e^{(-\frac{1}{2}a^2+\frac{1}{2})t} y(x, s; a)^{-1-\frac{1}{a}} (y(x, s; a)^2 + 1). \quad (\text{A.9})$$

Since  $y(x, s; a)$  is always positive and  $b > 0$ , it is easily verified that  $u_x(x, s) > 0$  and  $u_{xx}(x, s) < 0$  for all  $a > 0$  from (A.7) and (A.8). So we conclude that  $x \rightarrow U_t(x)$  is strictly concave and increasing for all  $a > 0$ . From (A.7), (A.8), and (A.9), we observe that  $u(x, s)$  satisfies  $u_s(x, s) = u_x^2(x, s)/2u_{xx}(x, s)$  for all  $a > 0$ ,  $x \in \mathbb{R}$ , and  $s \geq 0$ . In the proof of Proposition 3.1, we have already seen that this implies that

$$E(U_t(X_t^\pi) | \mathcal{F}_s) \leq U_s(X_s^\pi),$$

for any  $\pi \in \mathcal{A}$ . Now it is enough to show that, for  $0 \leq s \leq t$ ,

$$E(U_t(X_t^*)|\mathcal{F}_s) = U_s(X_s^*), \quad (\text{A.10})$$

where  $X_t^*$  is given by (9). For all  $a > 0$ , we have

$$y(X_t^*, A_t; a) = \left( \frac{a}{b}x + \sqrt{\frac{a^2}{b^2}x^2 + 1} \right) e^{aA_t + aM_t}. \quad (\text{A.11})$$

Using (A.11), we observe that  $U_t(X_t^*) = u(X_t^*, A_t)$  can be written as, for  $a = 1$ ,

$$U_t(X_t^*) = \frac{1}{2}M_t + k_1 e^{-2A_t - 2M_t} + k_2, \quad (\text{A.12})$$

and for  $a \neq 1$ ,

$$U_t(X_t^*) = k_3 e^{-\frac{1}{2}(a-1)^2 A_t + (a-1)M_t} - k_4 e^{-\frac{1}{2}(a+1)^2 A_t - (a+1)M_t}, \quad (\text{A.13})$$

where  $k_1, k_2, k_3$ , and  $k_4$  are given by

$$\begin{aligned} k_1 &= \frac{1}{4} \left( \frac{1}{b}x + \sqrt{\frac{1}{b^2}x^2 + 1} \right)^{-2}, \\ k_2 &= \frac{1}{2} \ln \left( \frac{1}{b}x + \sqrt{\frac{1}{b^2}x^2 + 1} \right) + \frac{1}{2} \ln b + \frac{1}{4}, \\ k_3 &= \frac{b^{1-\frac{1}{a}}}{2(a-1)} \left( \frac{a}{b}x + \sqrt{\frac{a^2}{b^2}x^2 + 1} \right)^{\frac{a-1}{a}}, \\ k_4 &= \frac{b^{1-\frac{1}{a}}}{2(a+1)} \left( \frac{a}{b}x + \sqrt{\frac{a^2}{b^2}x^2 + 1} \right)^{-\frac{a+1}{a}}. \end{aligned}$$

Note that  $k_1, k_2, k_3$ , and  $k_4$  are constant for given  $a > 0, b > 0$ , and  $x \in \mathbb{R}$ . By applying Itô's formula to (A.12) and (A.13), we have, for  $a = 1$ ,

$$dU_t(X_t^*) = \left( \frac{1}{2} - 2k_1 e^{-2A_t - 2M_t} \right) \lambda_t dW_t, \quad (\text{A.14})$$

and for  $a \neq 1$ ,

$$dU_t(X_t^*) = \left( k_3(a-1)e^{-\frac{1}{2}(a-1)^2 A_t + (a-1)M_t} + k_4(a+1)e^{-\frac{1}{2}(a+1)^2 A_t - (a+1)M_t} \right) \lambda_t dW_t. \quad (\text{A.15})$$

In (A.14) and (A.15), we observe that the drift of  $dU_t(X_t^*)$  is zero for all  $a > 0$ . Thus, for any  $a > 0$ , it is concluded that (A.10) holds for  $0 \leq s \leq t$  and this completes the proof.  $\square$

*Proof of Proposition 4.2.* The proof follows from a similar argument to that in Proposition 3.2.  $\square$

## Appendix B. Derivation of SAHARA utility function in Section 4

In Chen et al. (2011), absolute risk aversion is given by

$$A(x) = \frac{\alpha}{\sqrt{\beta^2 + x^2}}.$$

So  $\alpha = 1/a$  and  $\beta = c/a$  for  $a > 0$  and  $c > 0$  in our model.

If  $\alpha = 1$ ,  $\hat{U}(x)$  is given by

$$\hat{U}(x) = \frac{1}{2} \ln(x + \sqrt{\beta^2 + x^2}) + \frac{1}{2} \beta^{-2} x (\sqrt{\beta^2 + x^2} - x)$$

in Proposition 2.2 of Chen et al. (2011). Since  $\beta = c$ , it is straightforward that  $\hat{U}(x) = U(x)$  where  $U(x)$  is given by (6).

If  $\alpha \neq 1$ ,  $\hat{U}(x)$  is given by

$$\hat{U}(x) = -\frac{1}{\alpha^2 - 1} \left( x + \sqrt{\beta^2 + x^2} \right)^{-\alpha} \left( x + \alpha \sqrt{\beta^2 + x^2} \right)$$

in Proposition 2.2 of Chen et al. (2011). Using  $\alpha = 1/a$  and  $\beta = c/a$ ,  $\hat{U}(x)$  can be rewritten as follows,

$$\begin{aligned} \hat{U}(x) &= \frac{a^2}{a^2 - 1} \left( x + \sqrt{\frac{c^2}{a^2} + x^2} \right)^{-\frac{1}{a}} \left( x + \frac{1}{a} \sqrt{\frac{c^2}{a^2} + x^2} \right) \\ &= \frac{a^{\frac{1}{a}}}{a^2 - 1} \frac{a(a+1) (ax^2 + x\sqrt{a^2x^2 + c^2}) + c^2}{(ax + \sqrt{a^2x^2 + c^2})^{\frac{1}{a}+1}} \\ &= a^{\frac{1}{a}} U(x), \end{aligned}$$

where  $U(x)$  is given in (7).

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